Spatial Filtering

Background

- Filter term in “Digital image processing” is referred to the subimage
- There are others term to call subimage such as mask, kernel, template, or window
- The value in a filter subimage are referred as coefficients, rather than pixels.

Basics of Spatial Filtering

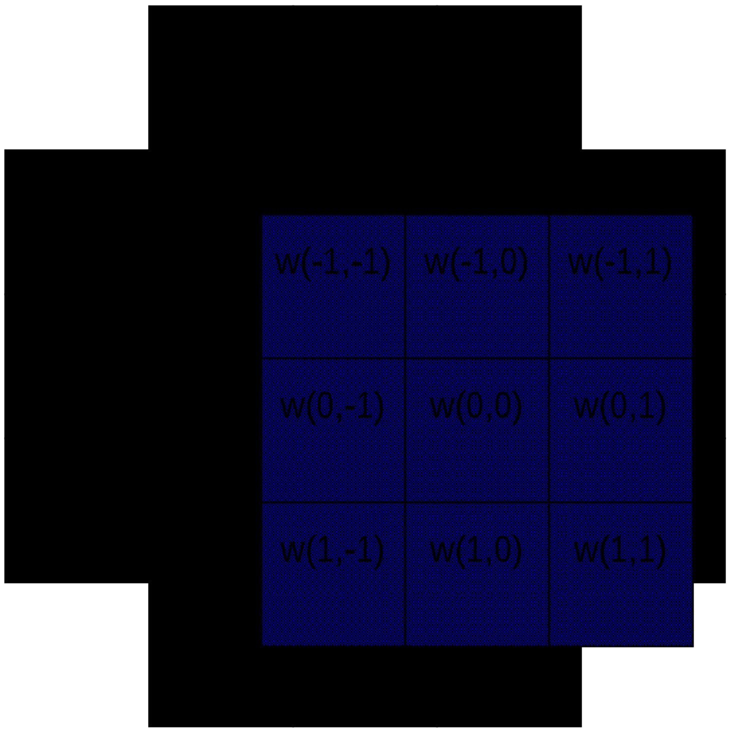
- The concept of filtering has its roots in the use of the Fourier transform for signal processing in the so-called frequency domain.
- Spatial filtering term is the filtering operations that are performed directly on the pixels of an image

Mechanics of spatial filtering

- The process consists simply of moving the filter mask from point to point in an image.
- At each point (x,y) the response of the filter at that point is calculated using a predefined relationship

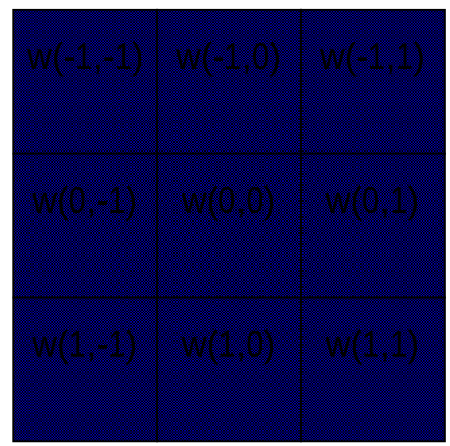
Linear spatial filtering

Pixels of image



The result is the sum of products of the mask coefficients with the corresponding pixels directly under the mask

Mask coefficients



$$
\begin{aligned}
f(x,y) = \; & w(-1,-1)\, f(x-1, y-1) + w(-1,0)\, f(x-1, y) + w(-1,1)\, f(x-1, y+1) + \\
& w(0,-1)\, f(x, y-1) + w(0,0)\, f(x, y) + w(0,1)\, f(x, y+1) + \\
& w(1,-1)\, f(x+1, y-1) + w(1,0)\, f(x+1, y) + w(1,1)\, f(x+1, y+1)
\end{aligned}
$$

Note: Linear filtering

- The coefficient $w(0,0)$ coincides with image value $f(x,y)$, indicating that the mask is centered at (x,y) when the computation of sum of products takes place.
- For a mask of size mxn, we assume that $m-2a+1$ and $n=2b+1$, where a and b are nonnegative integer. Then m and n are odd.

Linear filtering

- In general, linear filtering of an image f of size MxN with a filter mask of size mxn is given by the expression:

$$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t)$$

Discussion

- The process of linear filtering similar to a frequency domain concept called "*convolution*"

<u>Simplify expression</u>

$$R = w_1 z_1 + w_2 z_2 + \ldots + w_{mn} z_{mn} = \sum_{i=1}^{mn} w_i z_i$$

$$R = w_1 z_1 + w_2 z_2 + \ldots + w_9 z_9 = \sum_{i=1}^{9} w_i z_i$$

W_1	W_2	W_3
W_4	W_5	W_6
W_7	W_8	W_9

Where the w's are mask coefficients, the z's are the value of the image gray levels corresponding to those coefficients

Nonlinear spatial filtering

- Nonlinear spatial filters also operate on neighborhoods, and the mechanics of sliding a mask past an image are the same as was just outlined.
- The filtering operation is based conditionally on the values of the pixels in the neighborhood under consideration

Smoothing Spatial Filters

- Smoothing filters are used for blurring and for noise reduction.
  - Blurring is used in preprocessing steps, such as removal of small details from an image prior to object extraction, and bridging of small gaps in lines or curves
  - Noise reduction can be accomplished by blurring

Type of smoothing filtering

- There are 2 way of smoothing spatial filters
  - Smoothing Linear Filters
  - Order-Statistics Filters

Smoothing Linear Filters

- Linear spatial filter is simply the average of the pixels contained in the neighborhood of the filter mask.
- Sometimes called “averaging filters”.
- The idea is replacing the value of every pixel in an image by the average of the gray levels in the neighborhood defined by the filter mask.

Two 3x3 Smoothing Linear Filters

$$\frac{1}{9}\times\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Standard average

$$\frac{1}{16}\times\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix}$$

Weighted average

5x5 Smoothing Linear Filters

$$\frac{1}{25} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Smoothing Linear Filters

- The general implementation for filtering an MxN image with a weighted averaging filter of size mxn is given by the expression

$$g(x,y) = \frac{\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)\, f(x+s, y+t)}{\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)}$$

Result of Smoothing Linear Filters

Original Image

[3x3]

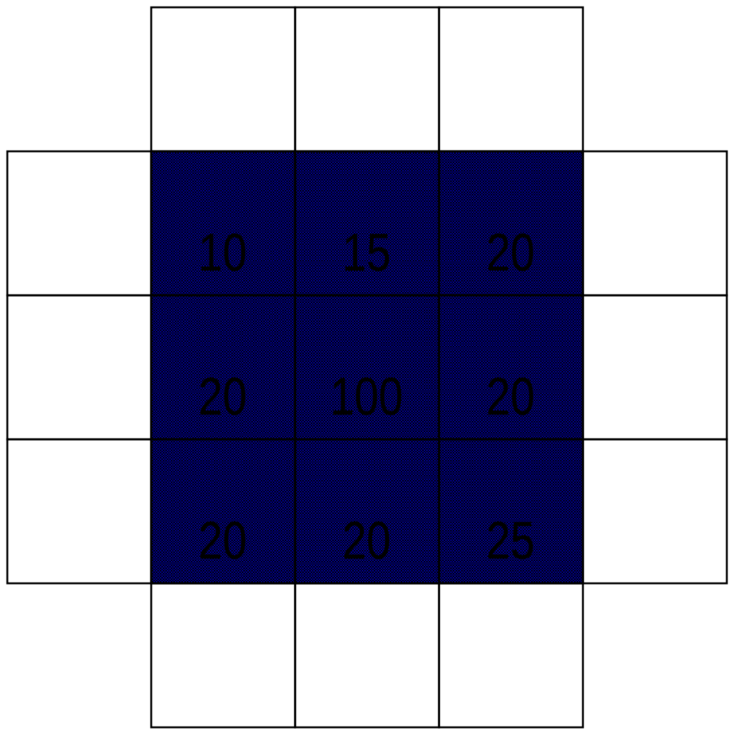
[5x5]

[7x7]

Order-Statistics Filters

- Order-statistics filters are nonlinear spatial filters whose response is based on ordering (ranking) the pixels contained in the image area encompassed by the filter, and then replacing the value of the center pixel with the value determined by the ranking result.
- Best-known “median filter”

Process of Median filter

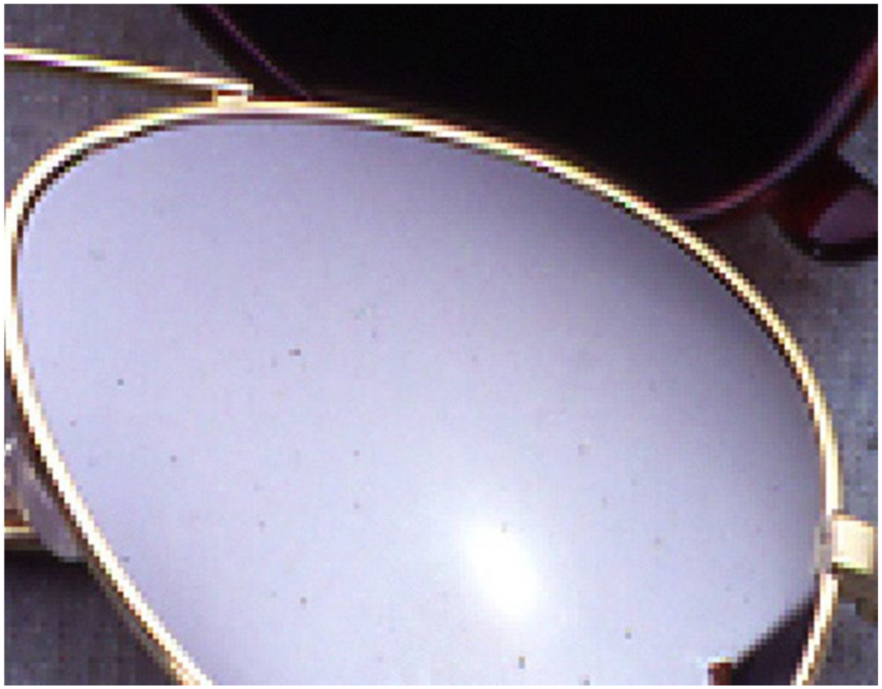


10, 15, 20, 20, 20, 20, 20, 25, 100

5^{th}

- Corp region of neighborhood
- Sort the values of the pixel in our region
- In the MxN mask the median is MxN div 2 +1

Result of median filter



Noise from Glass effect

Remove noise by median filter

Sharpening Spatial Filters

- The principal objective of sharpening is to highlight fine detail in an image or to enhance detail that has been blurred, either in error or as an natural effect of a particular method of image acquisition.

Introduction

- The image blurring is accomplished in the spatial domain by pixel averaging in a neighborhood.
- Since averaging is analogous to integration.
- Sharpening could be accomplished by spatial differentiation.

Foundation

- We are interested in the behavior of these derivatives in areas of constant gray level(flat segments), at the onset and end of discontinuities(step and ramp discontinuities), and along gray-level ramps.
- These types of discontinuities can be noise points, lines, and edges.

Definition for a first derivative

- Must be zero in flat segments
- Must be nonzero at the onset of a gray-level step or ramp; and
- Must be nonzero along ramps.

Definition for a second derivative

- Must be zero in flat areas;
- Must be nonzero at the onset and end of a gray-level step or ramp;
- Must be zero along ramps of constant slope

Definition of the 1st-order derivative

- A basic definition of the first-order derivative of a one-dimensional function f(x) is

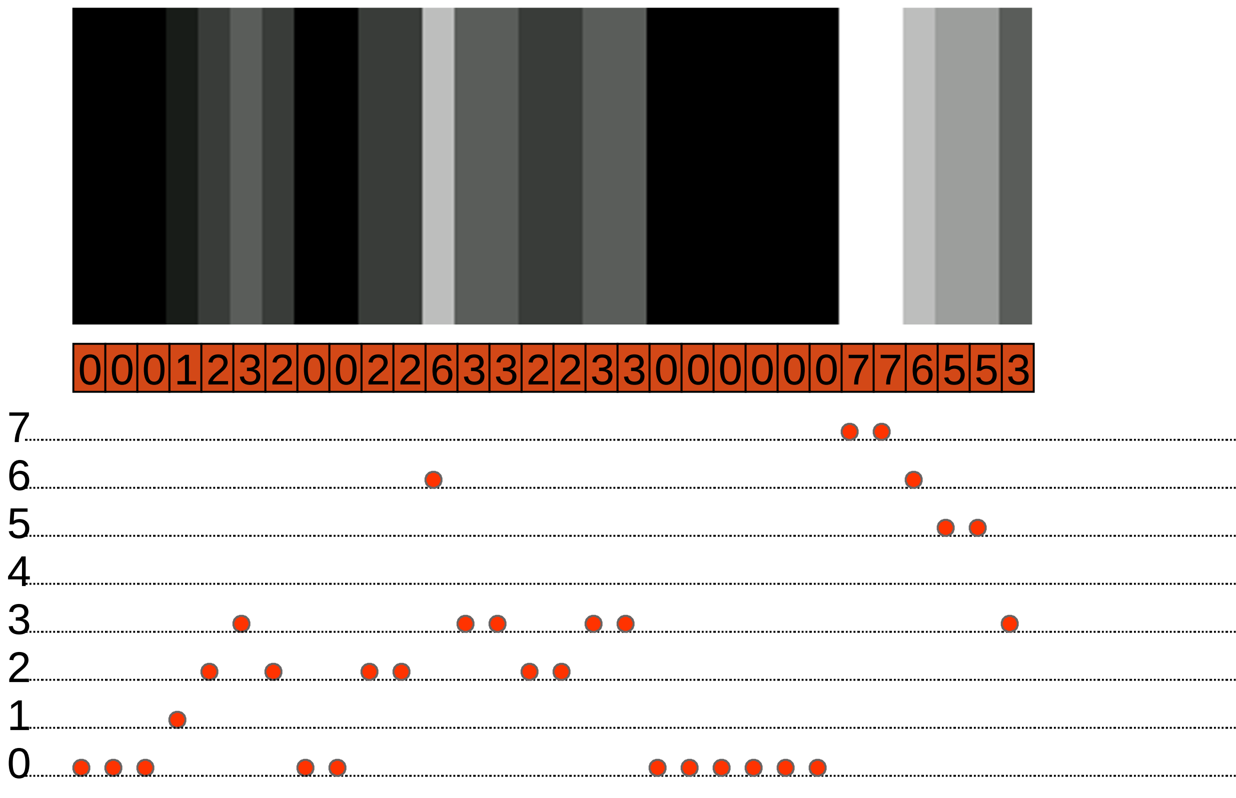
$$\frac{\partial f}{\partial x} = f(x+1) - f(x)$$

Definition of the 2nd-order derivative

- We define a second-order derivative as the difference

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x).$$

Gray-level profile



Derivative of image profile

0 0 0 1 2 3 2 0 0 2 2 6 3 3 2 2 3 3 0 0 0 0 0 0 7 7 6 5 5 3

first 0 0 1 1 1 -1 -2 0 2 0 4 -3 0 -1 0 1 0 -3 0 0 0 0 0 -7 0 -1 -1 0 -2

second 0 -1 0 0 -2 -1 2 2 -2 4 -7 3 -1 1 1 -1 -3 3 0 0 0 0 -7 7 -1 0 1 -2

Analyze

- The 1st-order derivative is nonzero along the entire ramp, while the 2nd-order derivative is nonzero only at the onset and end of the ramp.

- The response at and around the point is much stronger for the 2nd- than for the 1st-order derivative

1st make thick edge and 2nd make thin edge

The Laplacian (2nd order derivative)

- Shown by Rosenfeld and Kak[1982] that the simplest isotropic derivative operator is the Laplacian is defined as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Discrete form of derivative

f(x-1,y)	*f(x,y)*	*f(x+1,y)*

$$\frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y)$$

f(x,y-1)
f(x,y)
f(x,y+1)

$$\frac{\partial^2 f}{\partial y^2} = f(x, y+1) + f(x, y-1) - 2f(x, y)$$

2-Dimentional Laplacian

- The digital implementation of the 2-Dimensional Laplacian is obtained by summing 2 components

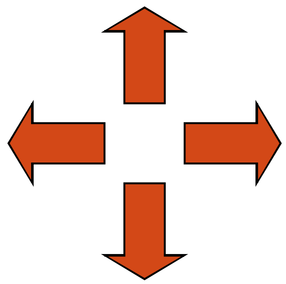
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}$$

$$\nabla^2 f = f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1) - 4f(x, y)$$

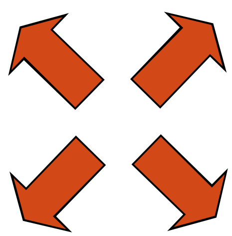
	1	
1	-4	1
	1	

Laplacian

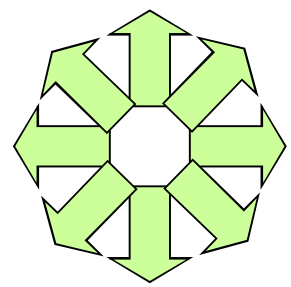
0	1	0
1	-4	1
0	1	0



1	0	1
0	-4	0
1	0	1



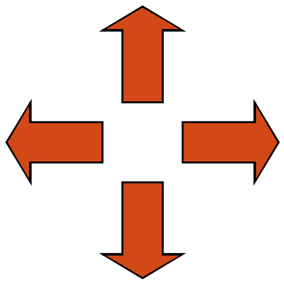
1	1	1
1	-8	1
1	1	1

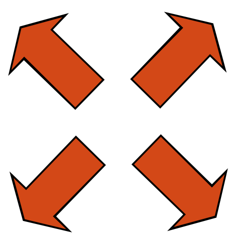


Laplacian

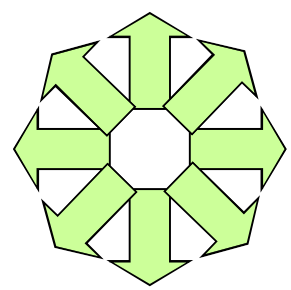
0	-1	0
-1	4	-1
0	-1	0

-1	0	-1
0	4	0
-1	0	-1





-1	-1	-1
-1	8	-1
-1	-1	-1



Implementation

$$g(x,y)=\begin{cases} f(x,y)-\nabla^2 f(x,y) & \textit{If the center coefficient is negative} \\ f(x,y)+\nabla^2 f(x,y) & \textit{If the center coefficient is positive} \end{cases}$$

Where f(x,y) is the original image

$\nabla^2 f(x,y)$ is Laplacian filtered image

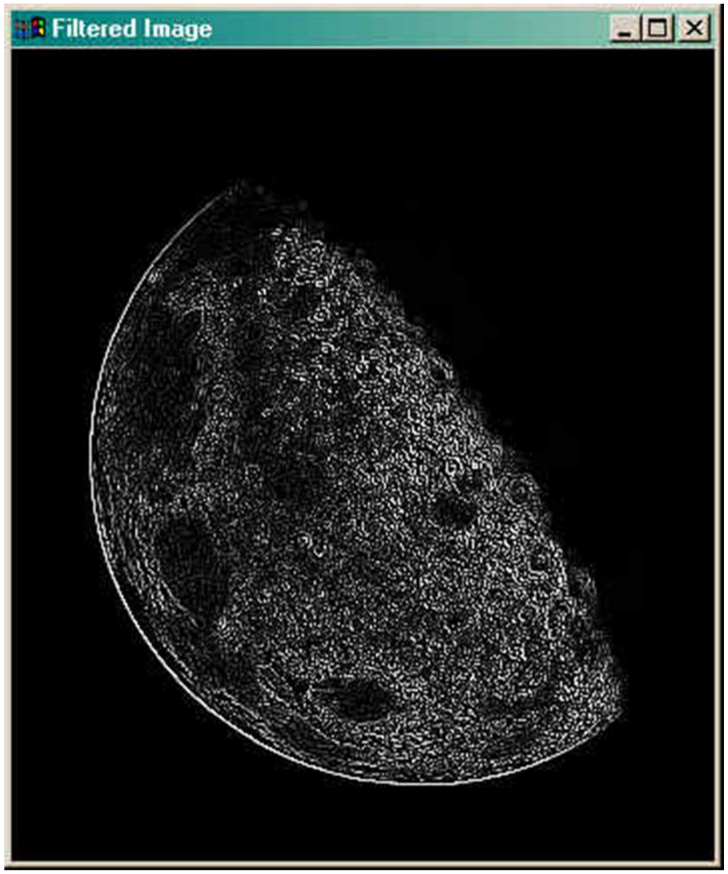
g(x,y) is the sharpen image

Implementation

-1	-1	-1
-1	8	-1
-1	-1	-1

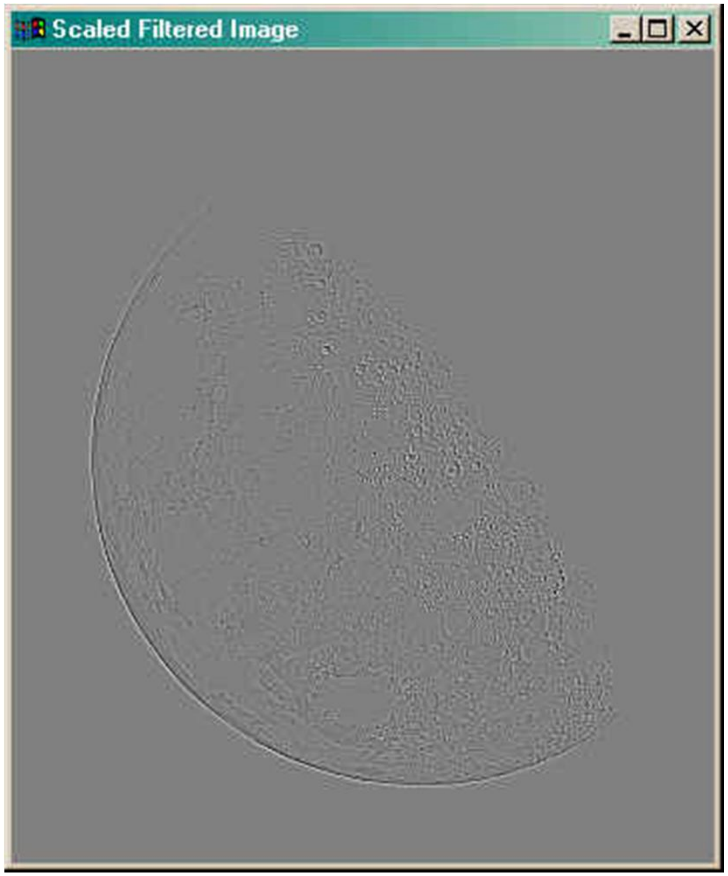
Implementation

Filtered = Conv(image,mask)



Implementation

```
filtered = filtered - Min(filtered)
filtered = filtered * (255.0 / Max(filtered))
```



Implementation

sharpened = image + filtered
sharpened = sharpened - Min(sharpened)
sharpened = sharpened * (255.0 / Max(sharpened))

Algorithm

- Using Laplacian filter to original image
- And then add the image result from step 1 and the original image

Simplification

- We will apply two step to be one mask

$$g(x,y) = f(x,y) - f(x+1,y) - f(x-1,y) - f(x,y+1) - f(x,y-1) + 4f(x,y)$$

$$g(x,y) = 5f(x,y) - f(x+1,y) - f(x-1,y) - f(x,y+1) - f(x,y-1)$$

0	-1	0
-1	5	-1
0	-1	0

-1	-1	-1
-1	9	-1
-1	-1	-1

Unsharp masking

- A process to sharpen images consists of subtracting a blurred version of an image from the image itself. This process, called *unsharp masking*, is expressed as

$$f_s(x,y) = f(x,y) - \bar{f}(x,y)$$

Where $f_s(x,y)$ denotes the sharpened image obtained by unsharp masking, and $\bar{f}(x,y)$ is a blurred version of $f(x,y)$

High-boost filtering

- A high-boost filtered image, f_{hb} is defined at any point (x,y) as

$$f_{hb}(x,y) = Af(x,y) - \bar{f}(x,y) \quad where\ A \geq 1$$

$$f_{hb}(x,y) = (A-1)f(x,y) + f(x,y) - \bar{f}(x,y)$$

$$f_{hb}(x,y) = (A-1)f(x,y) + f_s(x,y)$$

This equation is applicable general and does not state explicity how the sharp image is obtained

High-boost filtering and Laplacian

- If we choose to use the Laplacian, then we know $f_s(x,y)$

$$f_{hb} = \begin{cases} Af(x,y) - \nabla^2 f(x,y) & \textit{If the center coefficient is negative} \\ Af(x,y) + \nabla^2 f(x,y) & \textit{If the center coefficient is positive} \end{cases}$$

0	-1	0
-1	A+4	-1
0	-1	0

-1	-1	-1
-1	A+8	-1
-1	-1	-1

The Gradient (1^{st} order derivative)

- First Derivatives in image processing are implemented using the magnitude of the gradient.
- The gradient of function f(x,y) is

$$\nabla f = \begin{bmatrix} G_x \\ G_y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}$$

Gradient

- The magnitude of this vector is given by

$$mag(\nabla f) = \sqrt{G_x^2 + G_y^2} \approx |G_x| + |G_y|$$

G_x

-1	1

This mask is simple, and no isotropic. Its result only horizontal and vertical.

G_y

1
-1

Robert's Method

- The simplest approximations to a first-order derivative that satisfy the conditions stated in that section are

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

$G_x = (z_9 - z_5)$ and $G_y = (z_8 - z_6)$

$$\nabla f = \sqrt{(z_9 - z_5)^2 + (z_8 - z_6)^2}$$

$$\nabla f \approx |z_9 - z_5| + |z_8 - z_6|$$

Robert's Method

- These mask are referred to as the Roberts cross-gradient operators.

-1	0
0	1

0	-1
1	0

Sobel's Method

- Mask of even size are awkward to apply.
- The smallest filter mask should be 3x3.
- The difference between the third and first rows of the 3x3 mage region approximate derivative in x-direction, and the difference between the third and first column approximate derivative in y-direction.

Sobel's Method

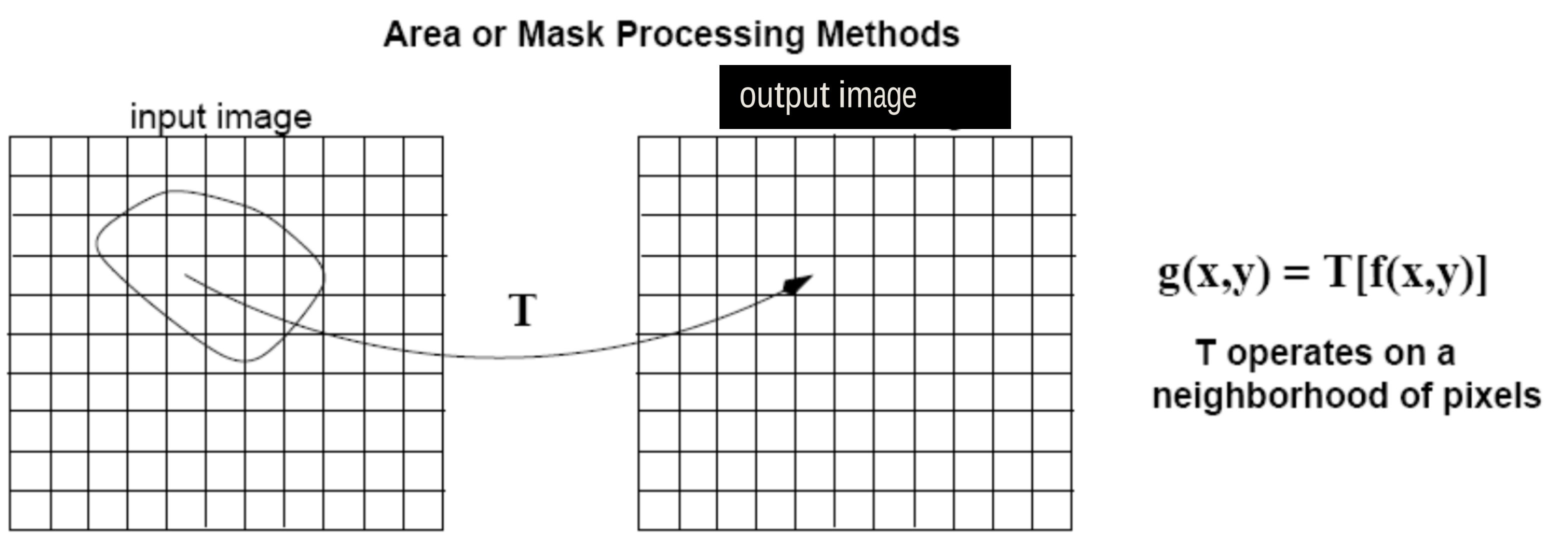
- Using this equation

$$\nabla f \approx \left|(z_7 + 2z_8 + z_9) - (z_1 + 2z_2 + z_3)\right| + \left|(z_3 + 2z_6 + z_9) - (z_1 + 2z_4 + z_7)\right|$$

-1	-2	-1
0	0	0
1	2	1

-1	0	1
-2	0	2
-1	0	1

Spatial Filtering Methods
(or Mask Processing Methods)



Spatial Filtering

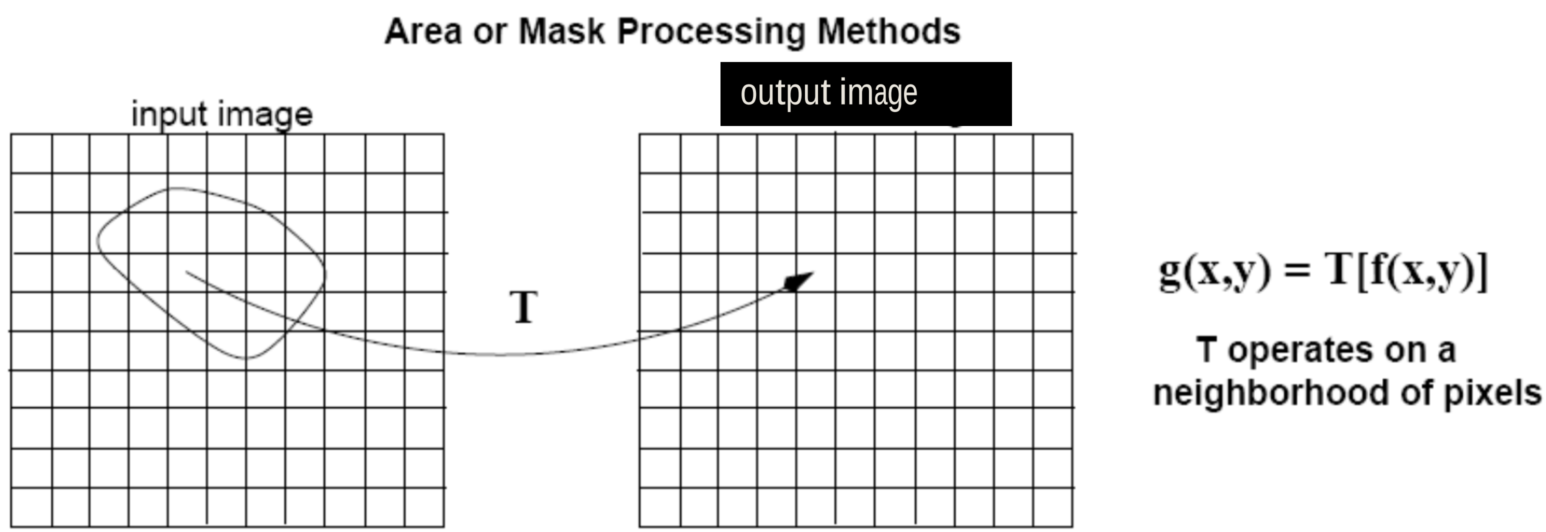
- The word “filtering” has been borrowed from the frequency domain.
- Filters are classified as:
  - Low-pass (i.e., preserve low frequencies)
  - High-pass (i.e., preserve high frequencies)
  - Band-pass (i.e., preserve frequencies within a band)
  - Band-reject (i.e., reject frequencies within a band)

Spatial Filtering (cont'd)

- Spatial filtering is defined by:

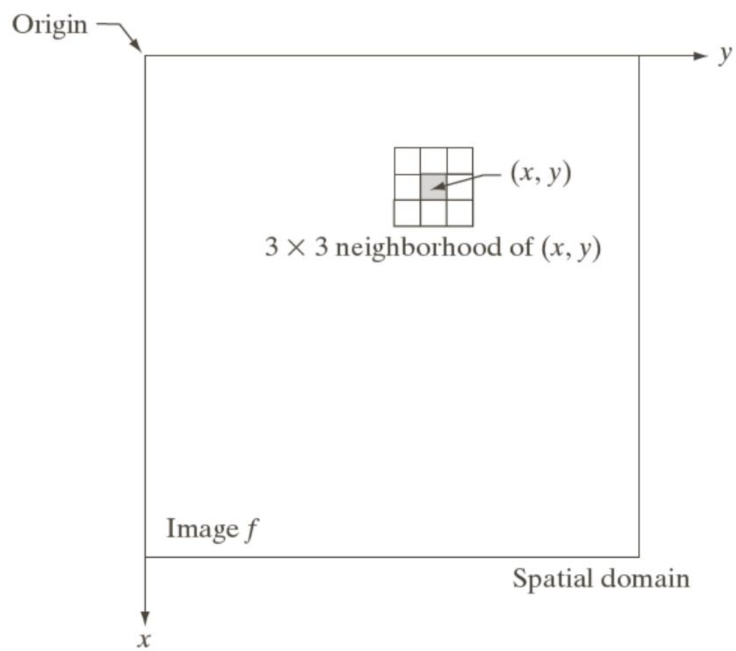
  (1) A neighborhood

  (2) An operation that is performed on the pixels inside the neighborhood

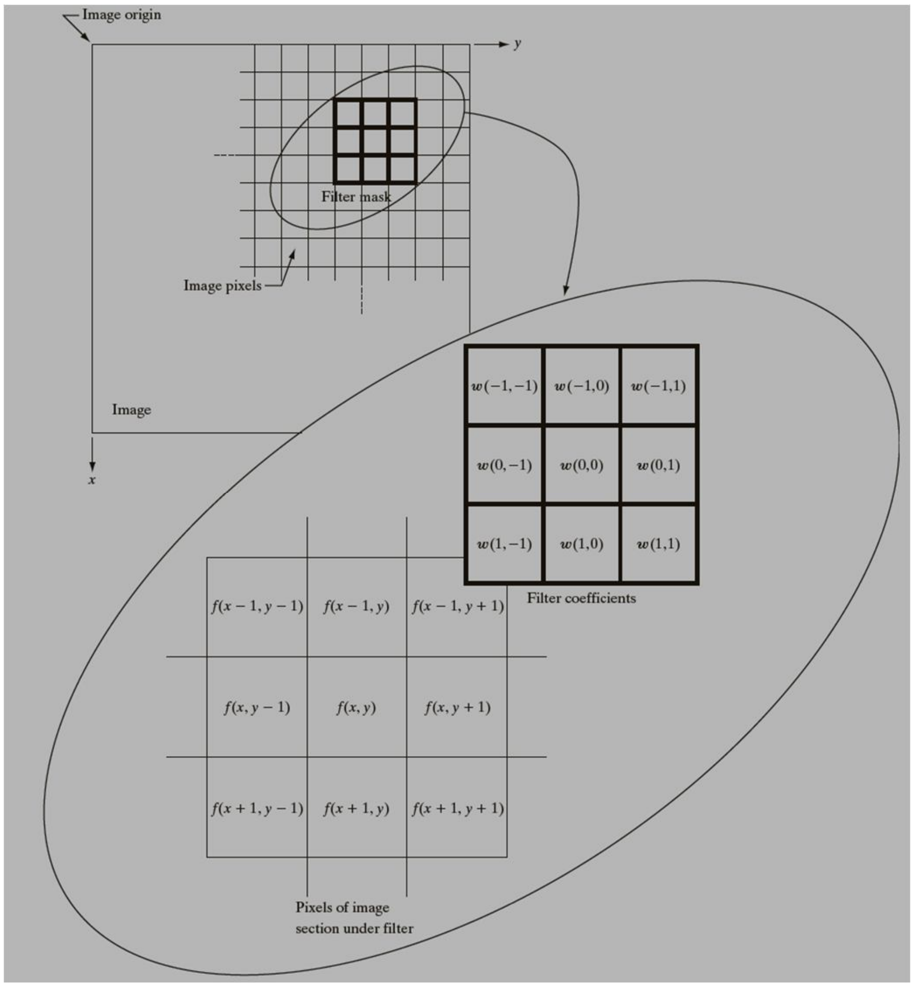


Spatial Filtering - Neighborhood

- Typically, the neighborhood is rectangular and its size is much smaller than that of *f(x,y)*
  - e.g., 3x3 or 5x5



Spatial filtering - Operation



Assume the origin of the mask is the center of the mask.

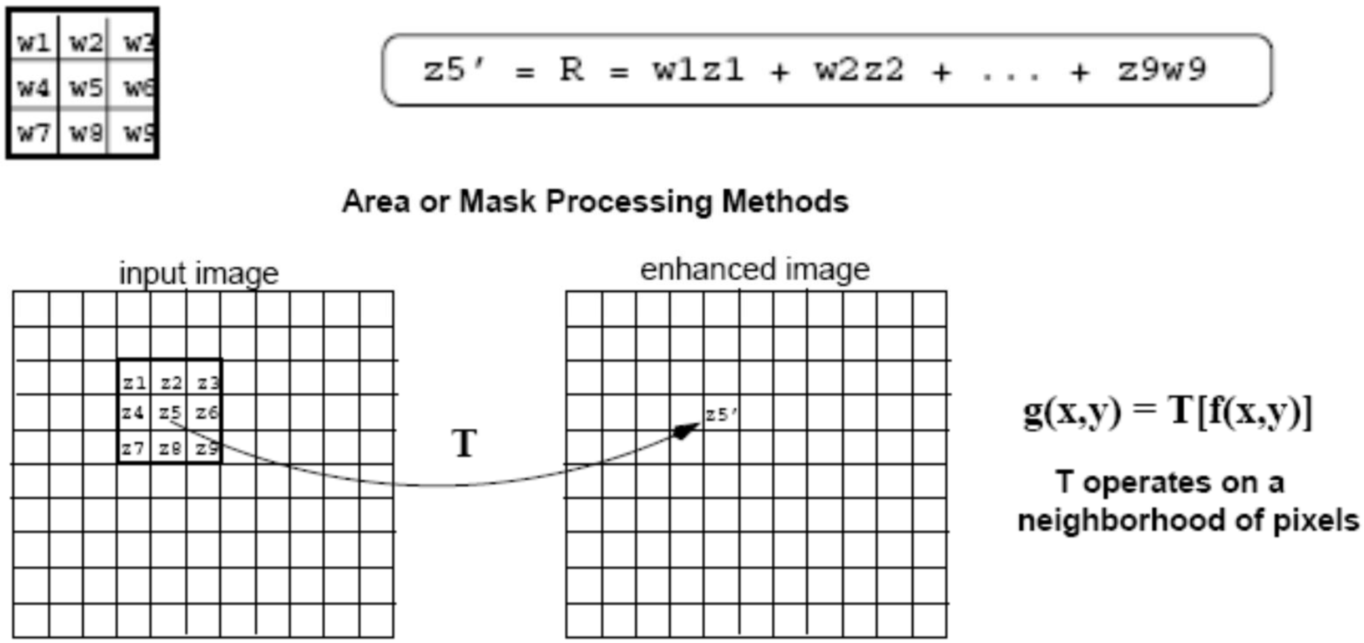
for a 3 x 3 mask:

$$g(x,y) = \sum_{s=-1}^{1} \sum_{t=-1}^{1} w(s,t) f(x+s, y+t)$$

for a K x K mask:

$$g(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x+s, y+t)$$

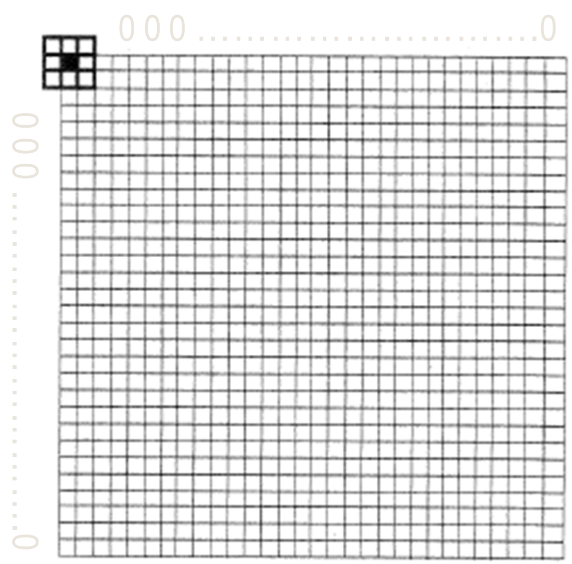
Spatial Filtering - Example



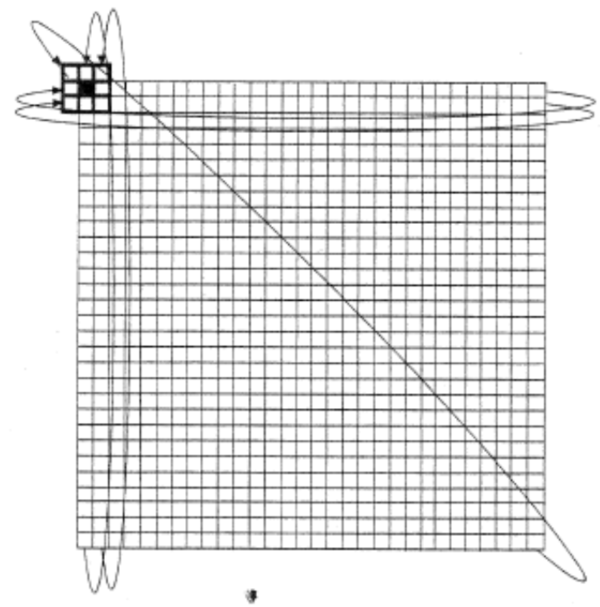
- A filtered image is generated as the center of the mask moves to every pixel in the input image.

Handling Pixels Close to Boundaries

pad with zeroes



or



Linear vs Non-Linear
Spatial Filtering Methods

- A filtering method is linear when the output is a weighted sum of the input pixels.

w1	w2	w3
w4	w5	w6
w7	w8	w9

$$z5' = R = w1z1 + w2z2 + \ldots + z9w9$$

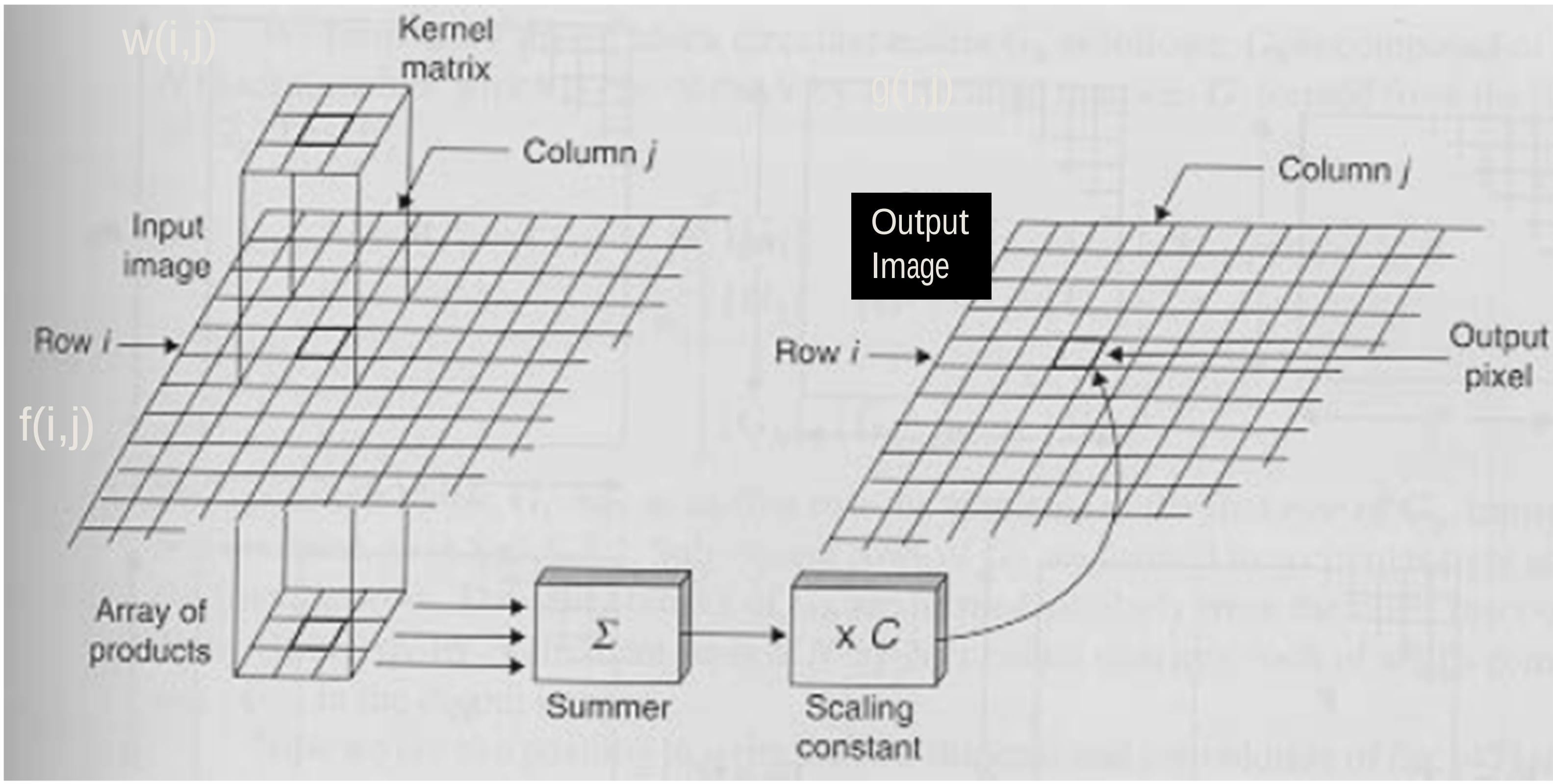
- Methods that do not satisfy the above property are called non-linear.
  - e.g.,

$$z_5' = max(z_k, k = 1, 2, \ldots, 9)$$

Linear Spatial Filtering Methods

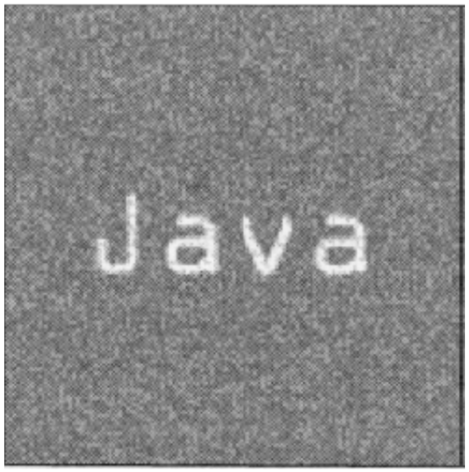
- Two main linear spatial filtering methods:
  - Correlation
  - Convolution

Correlation



$$g(x,y) = w(x,y) \bullet f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x+s, y+t)$$

Correlation (cont'd)



Often used in applications where we need to measure the similarity between images or parts of images
(e.g., pattern matching).

Convolution

- Similar to correlation except that the mask is first <u>**flipped**</u> both horizontally and vertically.

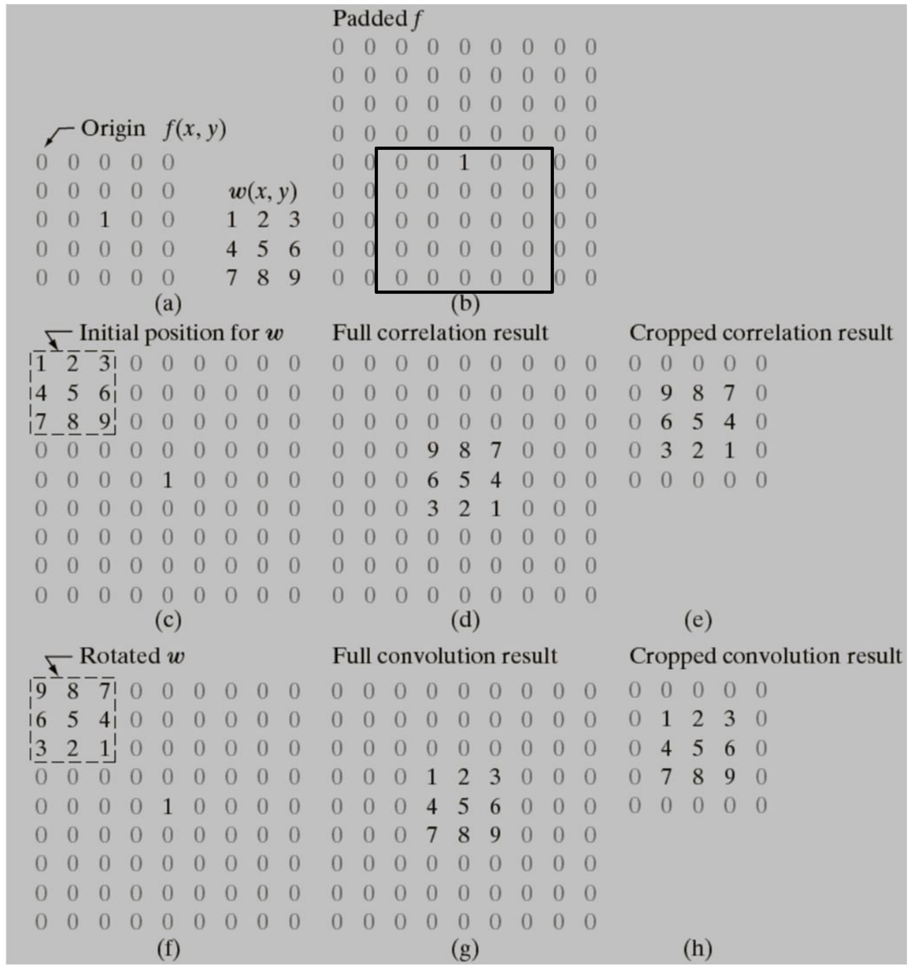
$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x-s, y-t)$$

<u>Note</u>: if w(x,y) is symmetric, that is w(x,y)=w(-x,-y), then convolution is equivalent to correlation!

Example

Correlation:

Convolution:



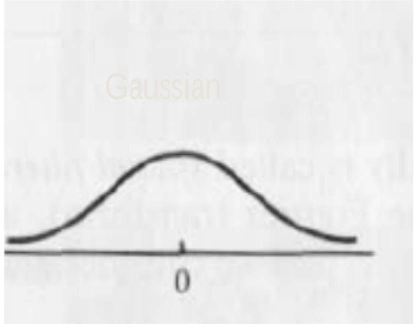
How do we choose the elements of a mask?

- Typically, by sampling certain functions.

w1	w2	w3
w4	w5	w6
w7	w8	w9

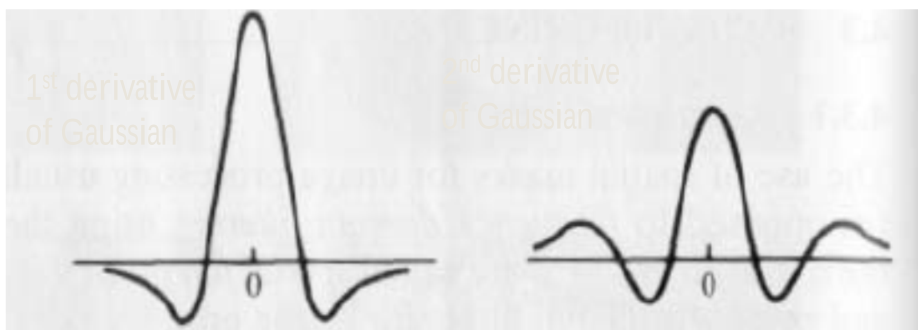
Filters

- Smoothing (i.e., low-pass filters)
  - Reduce noise and eliminate small details.
  - The elements of the mask must be **positive**.
  - Sum of mask elements is 1 (after normalization)

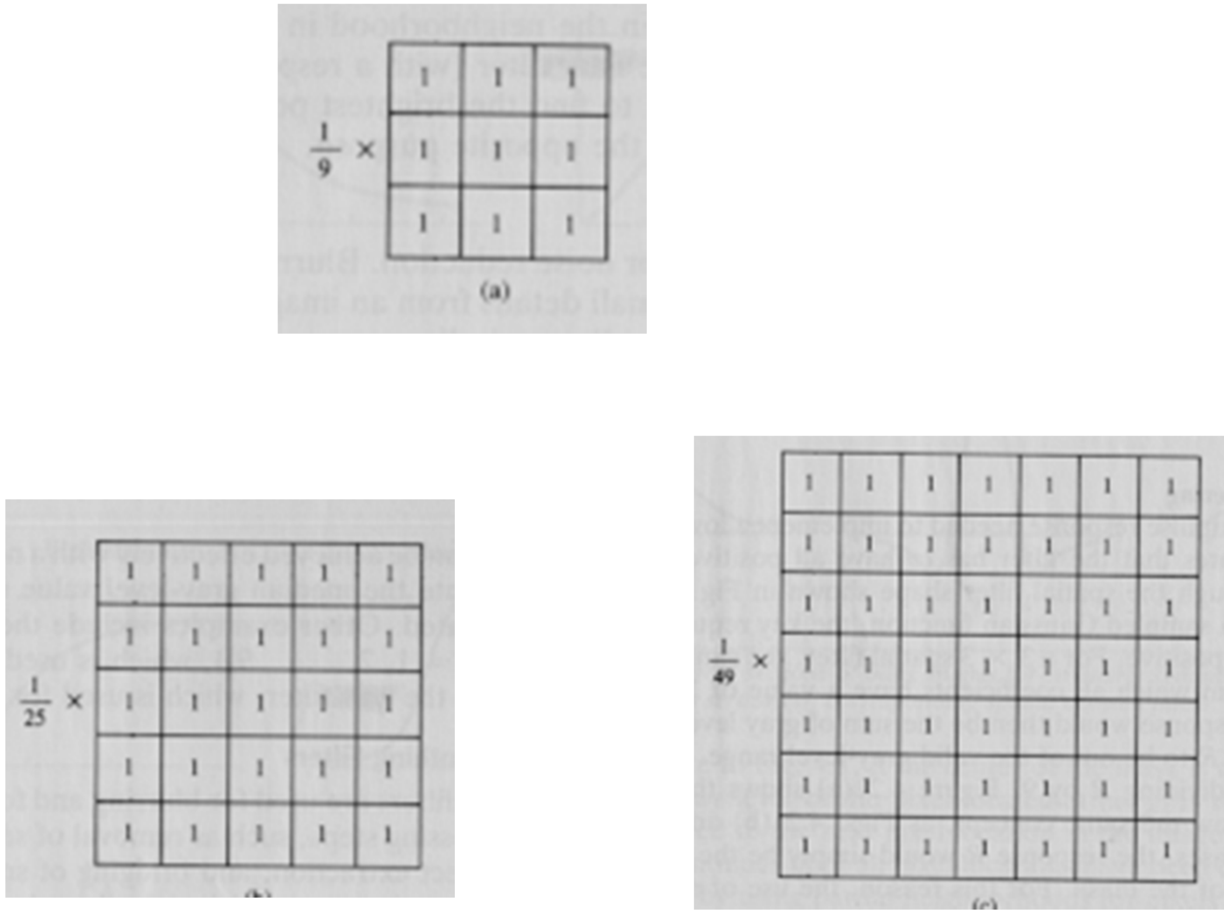


Filters (cont'd)

- Sharpening (i.e., high-pass filters)
  - Highlight fine detail or enhance detail that has been blurred.
  - The elements of the mask contain both **positive** and **negative** weights.
  - Sum of the mask weights is 0 (after normalization)



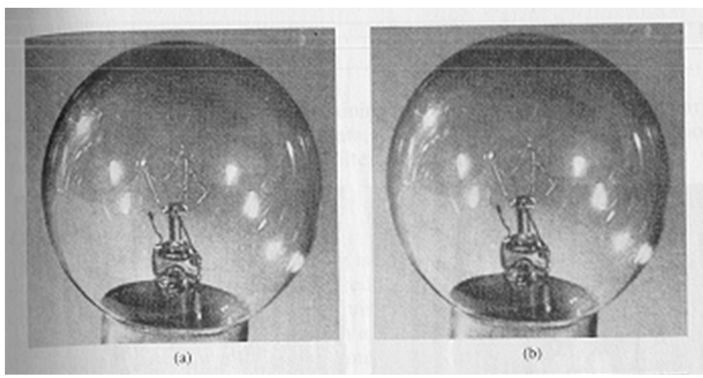
Smoothing Filters: Averaging (Low-pass filtering)



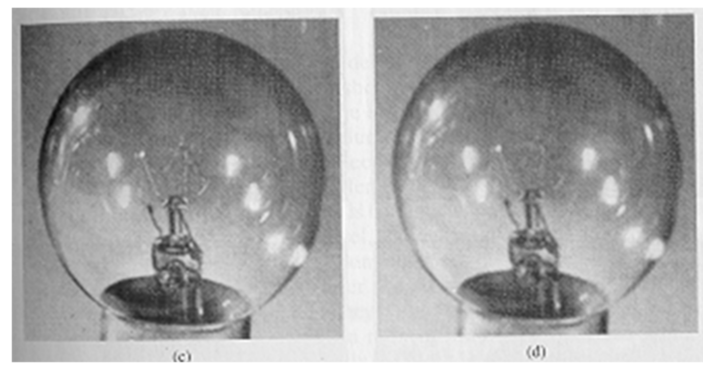
Smoothing Filters: Averaging (cont'd)

- Mask size determines the degree of smoothing and loss of detail.

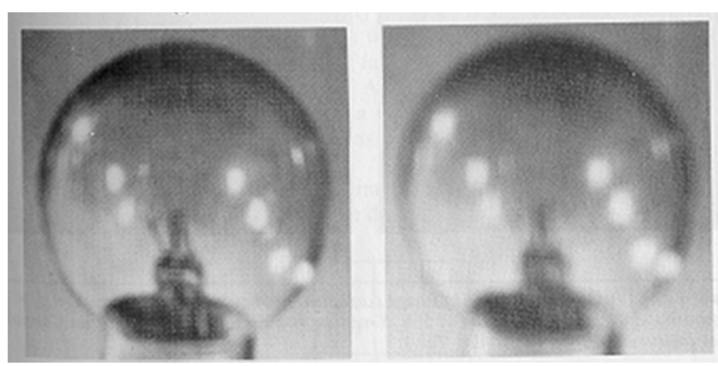
original 3x3



5x5 7x7

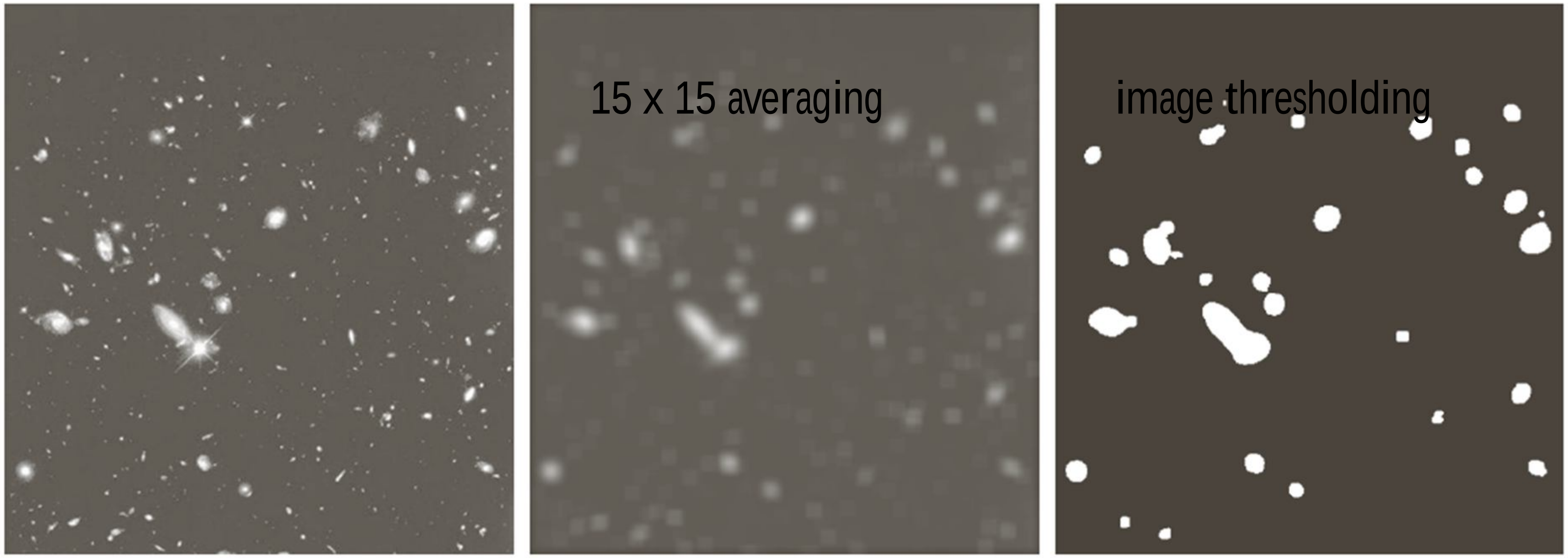


15x15 25x25



Smoothing Filters: Averaging (cont'd)

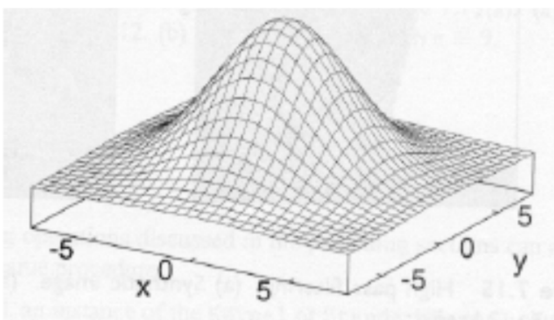
Example: extract, largest, brightest objects



Smoothing filters: Gaussian

- The weights are samples of the Gaussian function

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$



7 × 7 Gaussian mask

1	1	2	2	2	1	1
1	2	2	4	2	2	1
2	2	4	8	4	2	2
2	4	8	16	8	4	2
2	2	4	8	4	2	2
1	2	2	4	2	2	1
1	1	2	2	2	1	1

σ = 1.4

mask size is
a function of $\boldsymbol{\sigma}$:

$$height = width = 5\sigma \text{ (subtends 98.76\% of the area)}$$

Smoothing filters: Gaussian (cont'd)

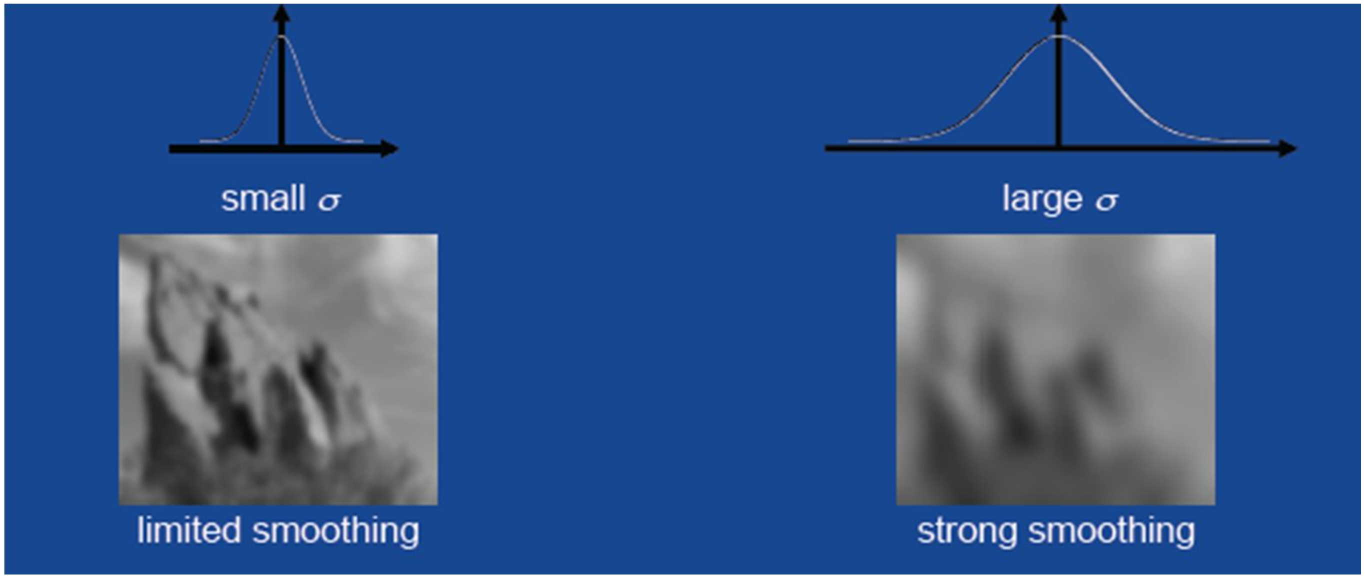
- σ controls the amount of smoothing
- As σ increases, more samples must be obtained to represent the Gaussian function accurately.

σ = 3

15 × 15 Gaussian mask

2	2	3	4	5	5	6	6	6	5	5	4	3	2	2
2	3	4	5	7	7	8	8	8	7	7	5	4	3	2
3	4	6	7	9	10	10	11	10	10	9	7	6	4	3
4	5	7	9	10	12	13	13	13	12	10	9	7	5	4
5	7	9	11	13	14	15	16	15	14	13	11	9	7	5
5	7	10	12	14	16	17	18	17	16	14	12	10	7	5
6	8	10	13	15	17	19	19	19	17	15	13	10	8	6
6	8	11	13	16	18	19	20	19	18	16	13	11	8	6
6	8	10	13	15	17	19	19	19	17	15	13	10	8	6
5	7	10	12	14	16	17	18	17	16	14	12	10	7	5
5	7	9	11	13	14	15	16	15	14	13	11	9	7	5
4	5	7	9	10	12	13	13	13	12	10	9	7	5	4
3	4	6	7	9	10	10	11	10	10	9	7	6	4	3
2	3	4	5	7	7	8	8	8	7	7	5	4	3	2
2	2	3	4	5	5	6	6	6	5	5	4	3	2	2

Smoothing filters: Gaussian (cont'd)



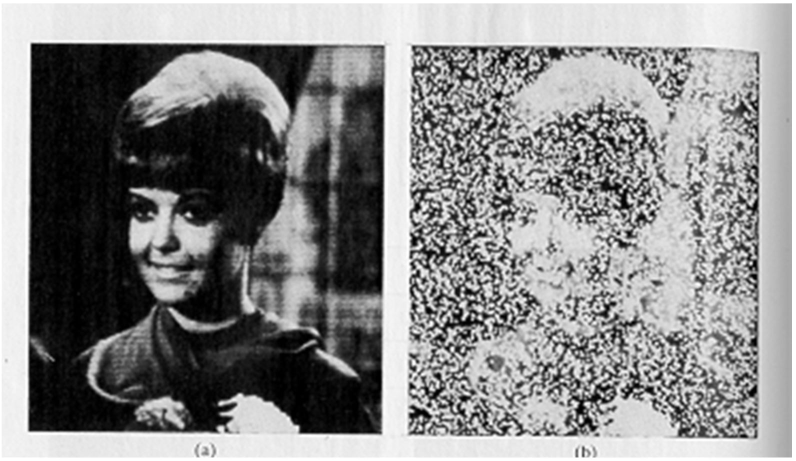
Averaging vs Gaussian Smoothing

Averaging

Gaussian

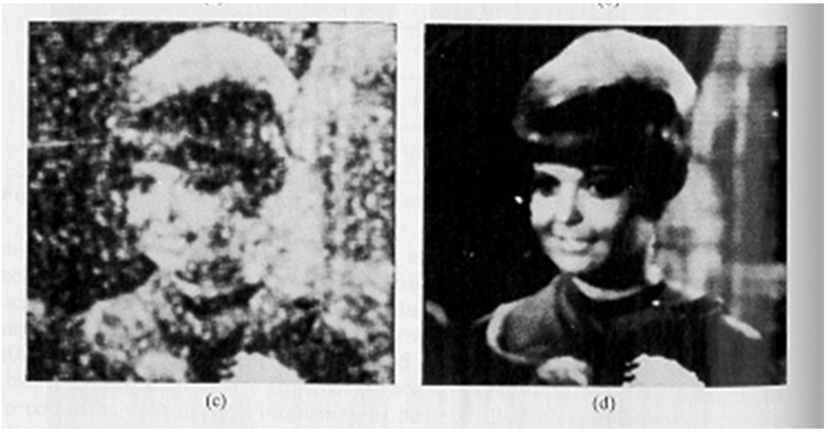
Smoothing Filters: Median Filtering (non-linear)

- Very effective for removing “salt and pepper” noise (i.e., random occurrences of black and white pixels).



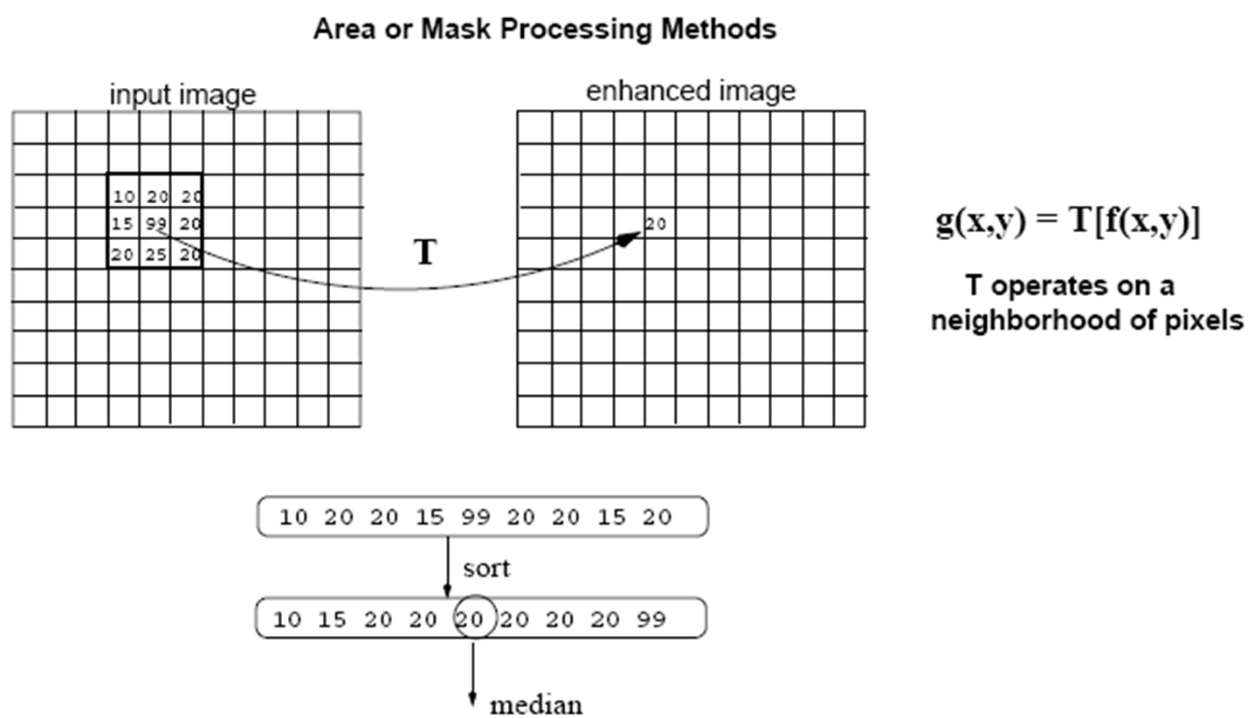
averaging

median filtering



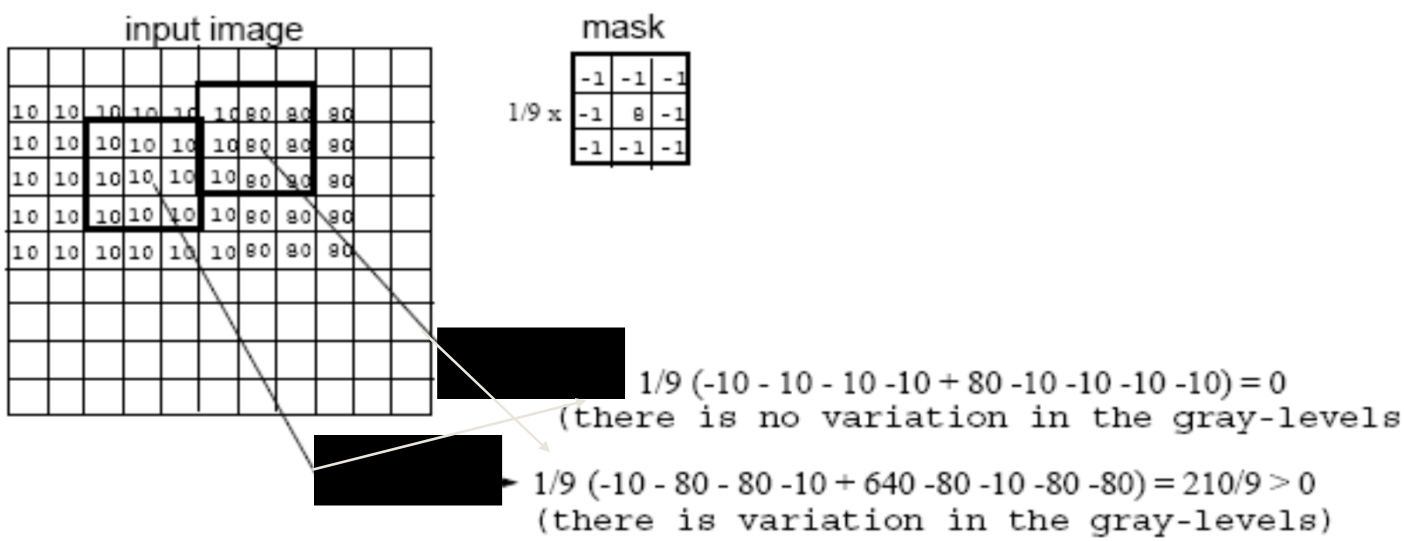
Smoothing Filters: Median Filtering (cont'd)

- Replace each pixel by the median in a neighborhood around the pixel.



Sharpening Filters (High Pass filtering)

- Useful for emphasizing transitions in image intensity (e.g., edges).



Sharpening Filters (cont'd)

- Note that the response of high-pass filtering might be negative.
- Values must be re-mapped to [0, 255]

original image

sharpened images

Sharpening Filters: Unsharp Masking

- Obtain a sharp image by subtracting a lowpass filtered (i.e., smoothed) image from the original image:

$$Highpass = Original - Lowpass$$

Sharpening Filters: High Boost

- Image sharpening emphasizes edges but details (i.e., low frequency components) might be lost.
- **High boost filter**: amplify input image, then subtract a lowpass image.

$$
\begin{aligned}
\textit{Highboost} &= A\ \textit{Original} - \textit{Lowpass} \\
&= (A-1)\ \textit{Original} + \textit{Original} - \textit{Lowpass} \\
&= (A-1)\ \textit{Original} + \textit{Highpass}
\end{aligned}
$$

(A-1)

Sharpening Filters: Unsharp Masking (cont'd)

- If **A=1**, we get a high pass filter

- If **A>1**, part of the original image is added back to the high pass filtered image.

A>=1
w = 9A-1

-1	-1	-1
-1	w	-1
-1	-1	-1

A=2
w = 17

-1	-1	-1
-1	17	-1
-1	-1	-1

Sharpening Filters: Unsharp Masking (cont'd)

A=1.4

A=1.9

Sharpening Filters: Derivatives

- Taking the derivative of an image results in sharpening the image.
- The derivative of an image can be computed using the gradient.

$$grad(f) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{pmatrix}$$

Sharpening Filters: Derivatives (cont'd)

- The gradient is a **vector** which has magnitude and direction:

$$magnitude(grad(f)) = \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}$$

or

$$\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$$

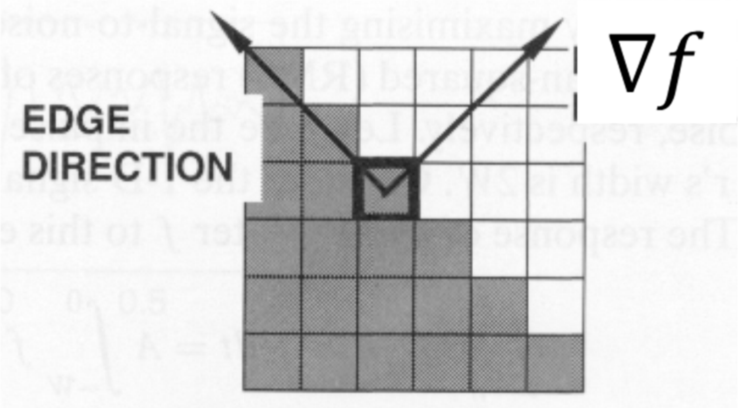
(approximation)

$$direction(grad(f)) = \tan^{-1}\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right)$$

Sharpening Filters: Derivatives (cont'd)

- **Magnitude:** provides information about edge strength.

- **Direction:** perpendicular to the direction of the edge.



Sharpening Filters: Gradient Computation

- Approximate gradient using finite differences:

$$f'(x) = \lim_{h->0} \frac{f(x+h) - f(x)}{h} \approx f(x+1) - f(x) \ \ (h{=}1)$$

sensitive to vertical edges

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = f(x+1,y) - f(x,y), \ \ (\Delta x{=}1)$$

sensitive to horizontal edges

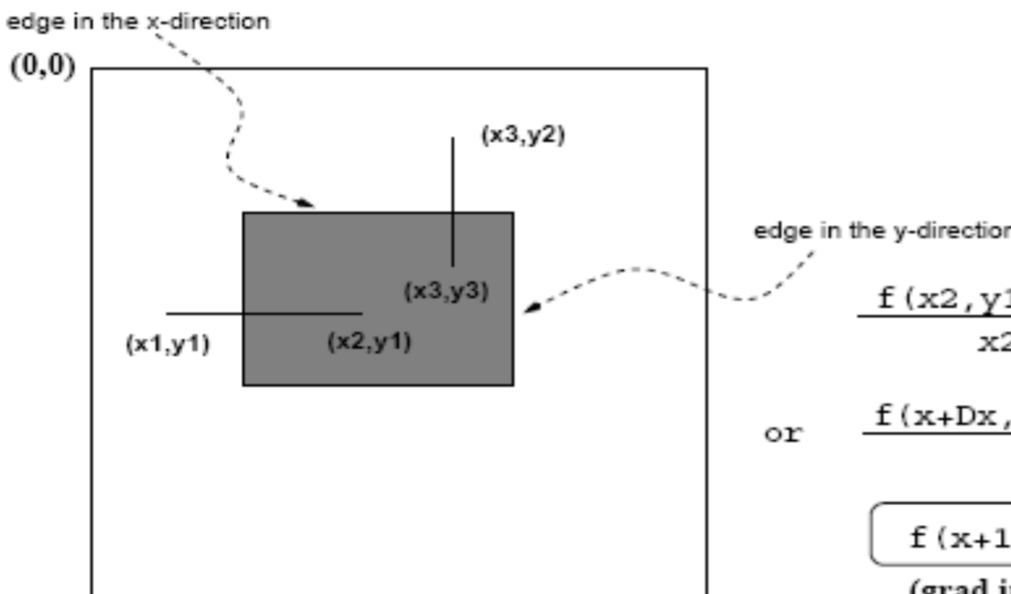
$$\frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x,y)}{-\Delta y} = f(x,y) - f(x,y+1), \ \ (\Delta y{=}1)$$

Sharpening Filters: Gradient Computation (cont'd)

$$\frac{f(x3,y2) - f(x3,y3)}{y2 - y3}$$

or $$\frac{f(x,y+Dy) - f(x,y)}{-Dy} = \boxed{f(x,y) - f(x,y+1)}$$ **(grad in the y-direction)**

$(y3=y2+Dy,\ y2=y,\ x3=x,\ Dy=1)$



$$\frac{f(x2,y1) - f(x1,y1)}{x2 - x1}$$

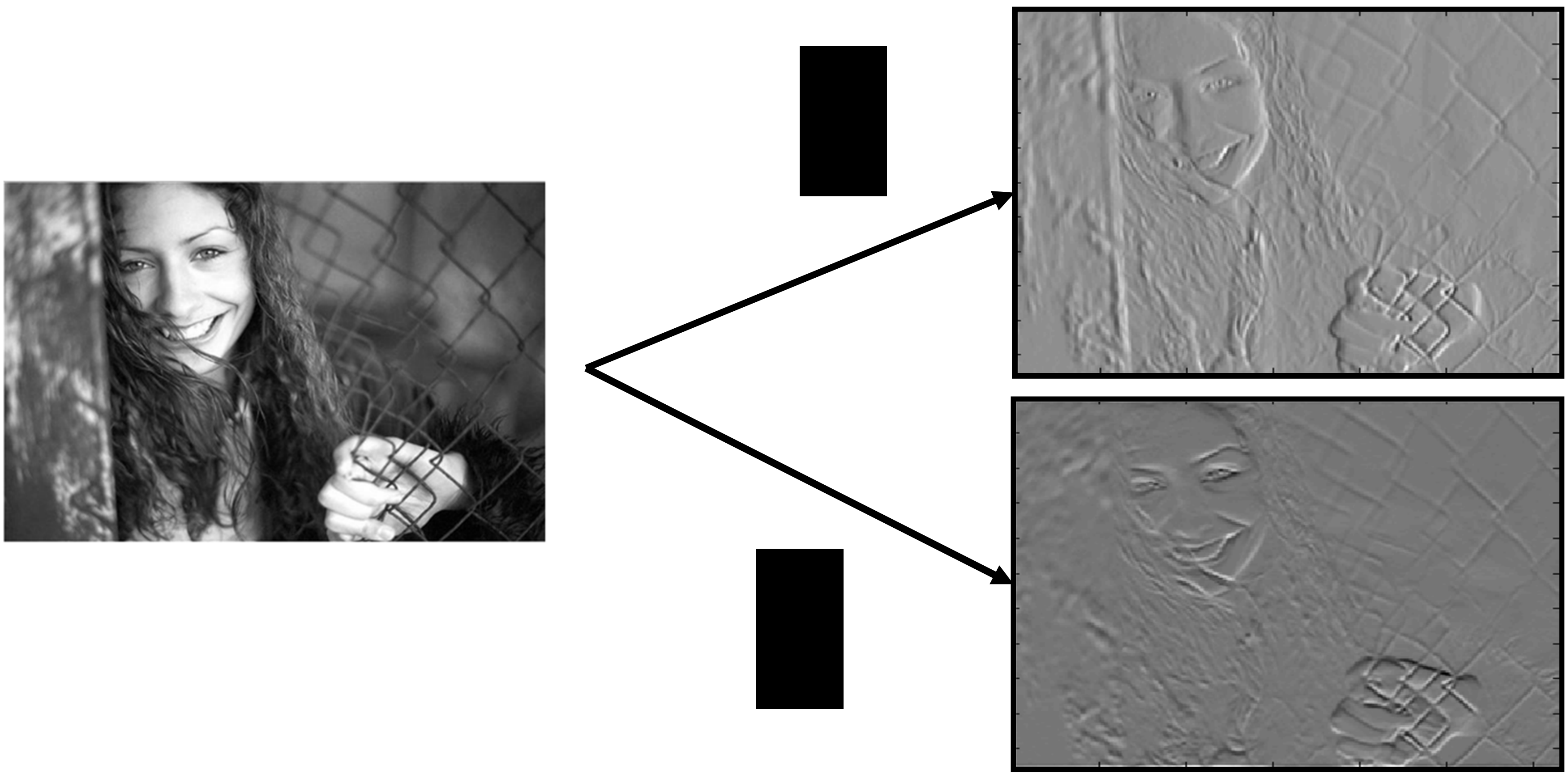
or $$\frac{f(x+Dx,y) - f(x,y)}{Dx} =$$

$$\boxed{f(x+1,y) - f(x,y)}$$

(grad in the x-direction)

$(x2=x+Dx,\ x1=x,\ y1=y2=y,\ Dx=1)$

Example



Sharpening Filters: Gradient Computation (cont'd)

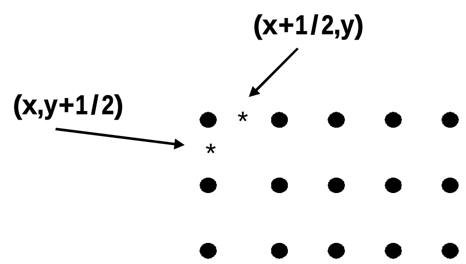
- We can implement $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ using masks:

-1	1

1
-1

$\dfrac{\partial f}{\partial x}$ good approximation at (x+1/2,y)

$\dfrac{\partial f}{\partial y}$ good approximation at (x,y+1/2)



- Example: approximate gradient at z_5

Z1	Z2	Z3
Z4	Z5	Z6
Z7	Z8	Z9

$$\frac{\partial f}{\partial x} = z_6 - z_5$$

$$\frac{\partial f}{\partial y} = z_5 - z_8$$

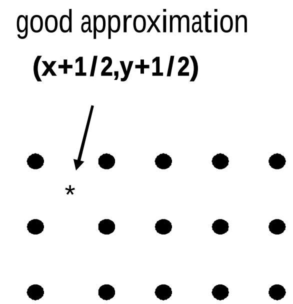
$$mag(grad(f)) = \sqrt{(z_6 - z_5)^2 + (z_5 - z_8)^2}$$

Sharpening Filters: Gradient Computation (cont’d)

- A different approximation of the gradient:

$$\frac{\partial f}{\partial x}(x, y) = f(x, y) - f(x+1, y+1)$$

$$\frac{\partial f}{\partial y}(x, y) = f(x+1, y) - f(x, y+1),$$



•We can implement and $\frac{\partial f}{\partial x}$ ing the $\frac{\partial f}{\partial y}$ ving masks:

1	0
0	–1

0	1
–1	0

(b) Roberts

Sharpening Filters: Gradient Computation (cont'd)

- <u>Example:</u> approximate gradient at z_5

z1	z2	z3
z4	z5	z6
z7	z8	z9

$$\frac{\partial f}{\partial x} = z_5 - z_9$$

$$\frac{\partial f}{\partial y} = z_6 - z_8$$

$$mag(grad(f)) = \sqrt{(z_5 - z_9)^2 + (z_6 - z_8)^2}$$

- Other approximations

–1	–1	–1
0	0	0
1	1	1

–1	0	1
–1	0	1
–1	0	1

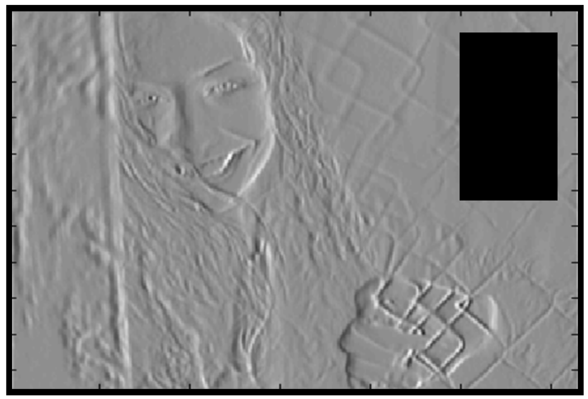
(c) Prewitt

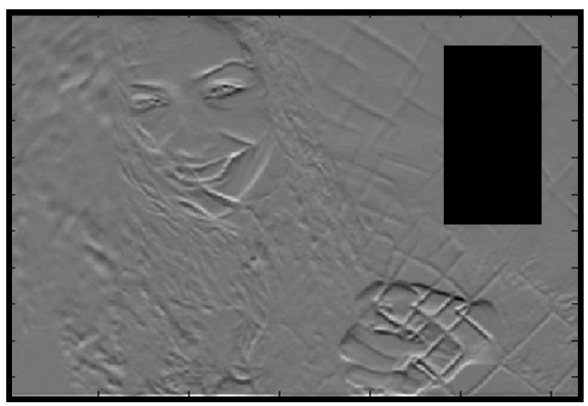
–1	–2	–1
0	0	0
1	2	1

–1	0	1
–2	0	2
–1	0	1

Sobel

Example





$$\sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}$$

Sharpening Filters: Laplacian

The Laplacian (2nd derivative) is defined as:

$$\nabla^2 = \nabla \cdot \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(dot product)

Approximate derivatives:

$$\frac{\partial^2 f}{\partial x^2} = f(i, j+1) - 2f(i, j) + f(i, j-1)$$

$$\frac{\partial^2 f}{\partial y^2} = f(i+1, j) - 2f(i, j) + f(i-1, j)$$

$$\nabla^2 f = -4f(i, j) + f(i, j+1) + f(i, j-1) + f(i+1, j) + f(i-1, j)$$

Sharpening Filters: Laplacian (cont'd)

Laplacian Mask

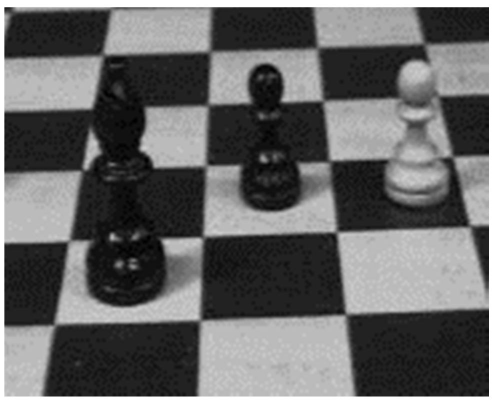
$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5	5	5	5	5	5
5	5	5	5	5	5
5	5	10	10	10	10
5	5	10	10	10	10
5	5	5	10	10	10
5	5	5	5	10	10

detect zero-crossings

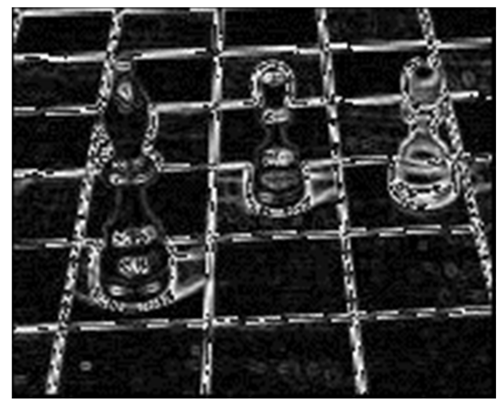
-	-	-	-	-	-
-	0	-5	-5	-5	-
-	-5	10	5	5	-
-	-5	10	0	0	-
-	0	-10	10	0	-
-	-	-	-	-	-

Sharpening Filters: Laplacian (cont'd)



Laplacian

Sobel



Neighborhood Processing (filtering)

2D filtering

- Cross-correlation in which the filter is flipped horizontally and vertically is called convolution

$$g = h * f$$

$$g[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} h[-u,-v] \cdot f[i+u, j+v]$$

$$\boxed{= \sum_{u=-k}^{k} \sum_{v=-k}^{k} h[u,v] \cdot f[i-u, j-v]}$$

Convolution vs. Cross-Correlation

- If the kernel is symmetric

$$h(u,v) = h(-u,-v)$$

convolution = cross-correlation

Neighborhood Processing
2D filtering

Noise

- **Types of noise:**
  - Salt and pepper noise
  - Impulse noise
  - Gaussian noise
- **Due to**
  - transmission errors
  - dead CCD pixels
  - specks on lens
  - can be specific to a sensor

Original

Salt and pepper noise

Impulse noise

Gaussian noise

Practical Noise Reduction

- How can we remove noise?
- Replace each pixel with the average of a kxk window around it

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	100	130	110	120	110	0	0
0	0	0	110	90	100	90	100	0	0
0	0	0	130	100	104	130	110	0	0
0	0	0	120	100	130	110	120	0	0
0	0	0	90	110	80	120	100	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$f[x, y]$

$g[x, y]$

Neighborhood Processing (filtering)

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

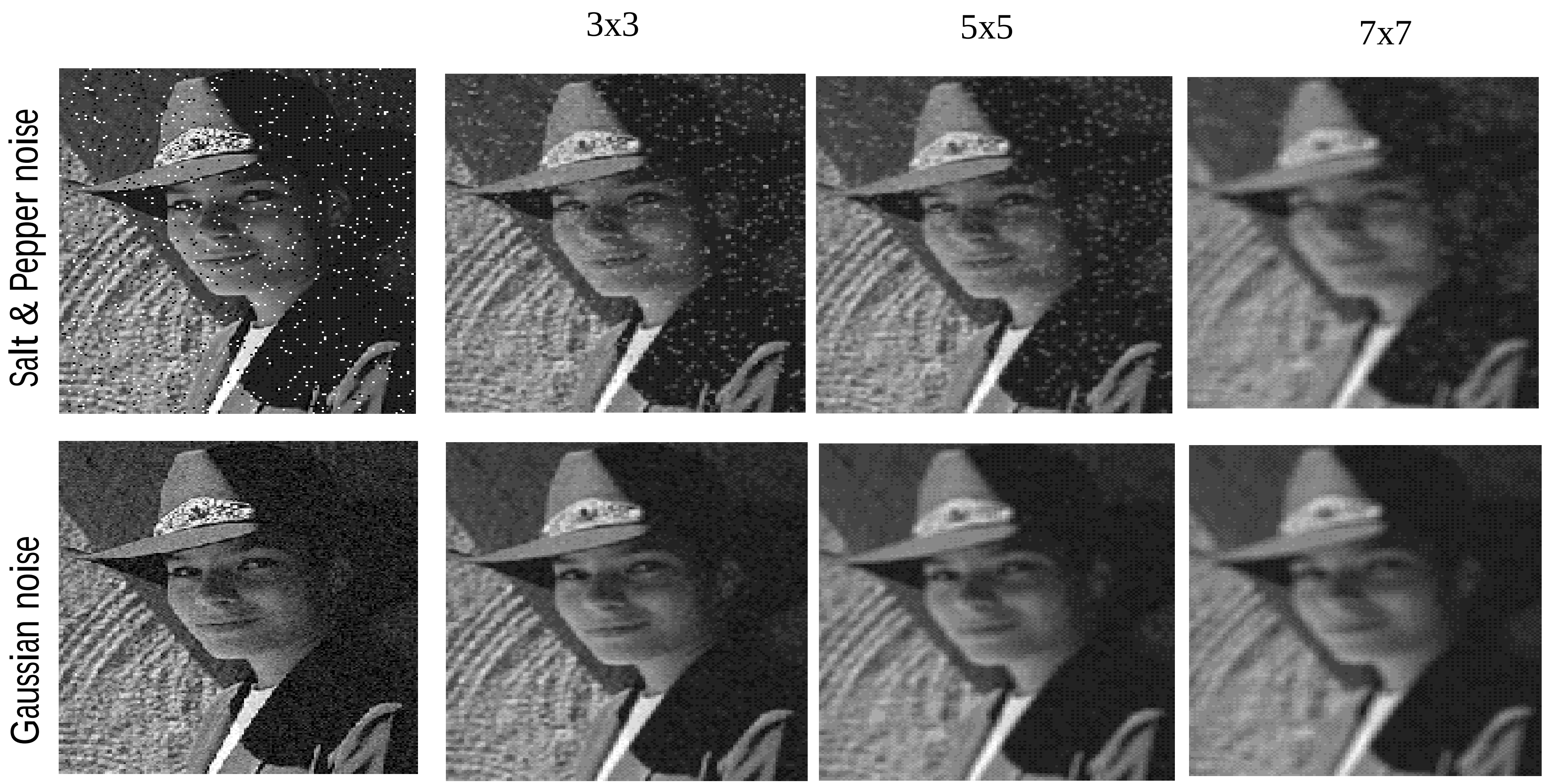
$f[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$g[x, y]$

Neighborhood Processing (filtering)

Effect of mean filters



Side effect - blur

Mean kernel

- What's the kernel for a 3x3 mean filter?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$f[x, y]$$

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$$h[u, v]$$

Mean kernel

- What's the kernel for a 3x3 mean filter?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$f[x,y]$

1/9

1	1	1
1	1	1
1	1	1

$h[u,v]$

Equal weight to all pixels within the neighborhood

Gaussian Filtering

- A **Gaussian kernel** gives less weight to pixels further from the center

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$f[x, y]$

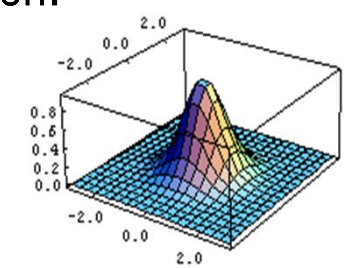
$\frac{1}{16} \cdot$

1	2	1
2	4	2
1	2	1

$h[u, v]$

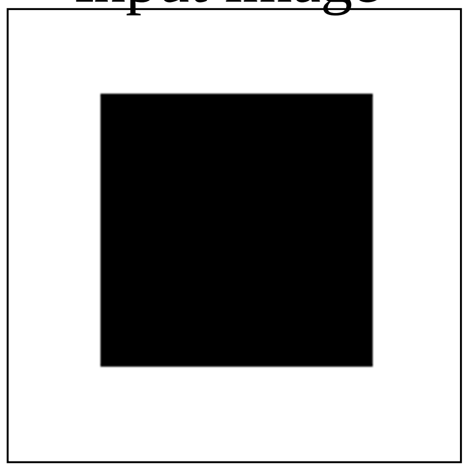
is a discrete approximation of a Gaussian function:

$$h(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{\sigma^2}}$$

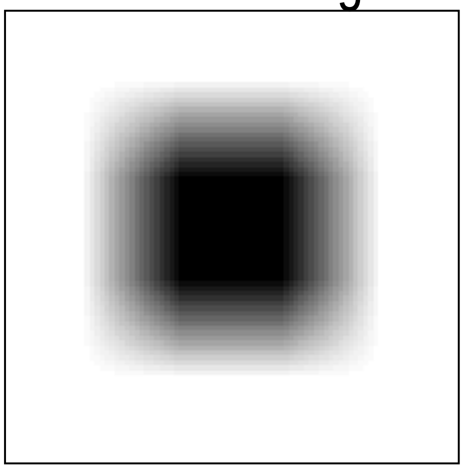


Mean vs. Gaussian filtering

Input image

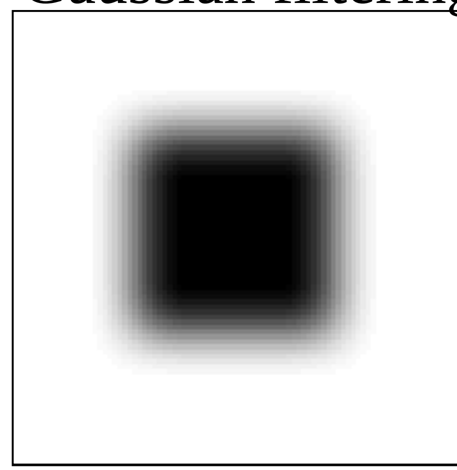


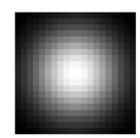
Mean Filtering



Mean filter, 20x20

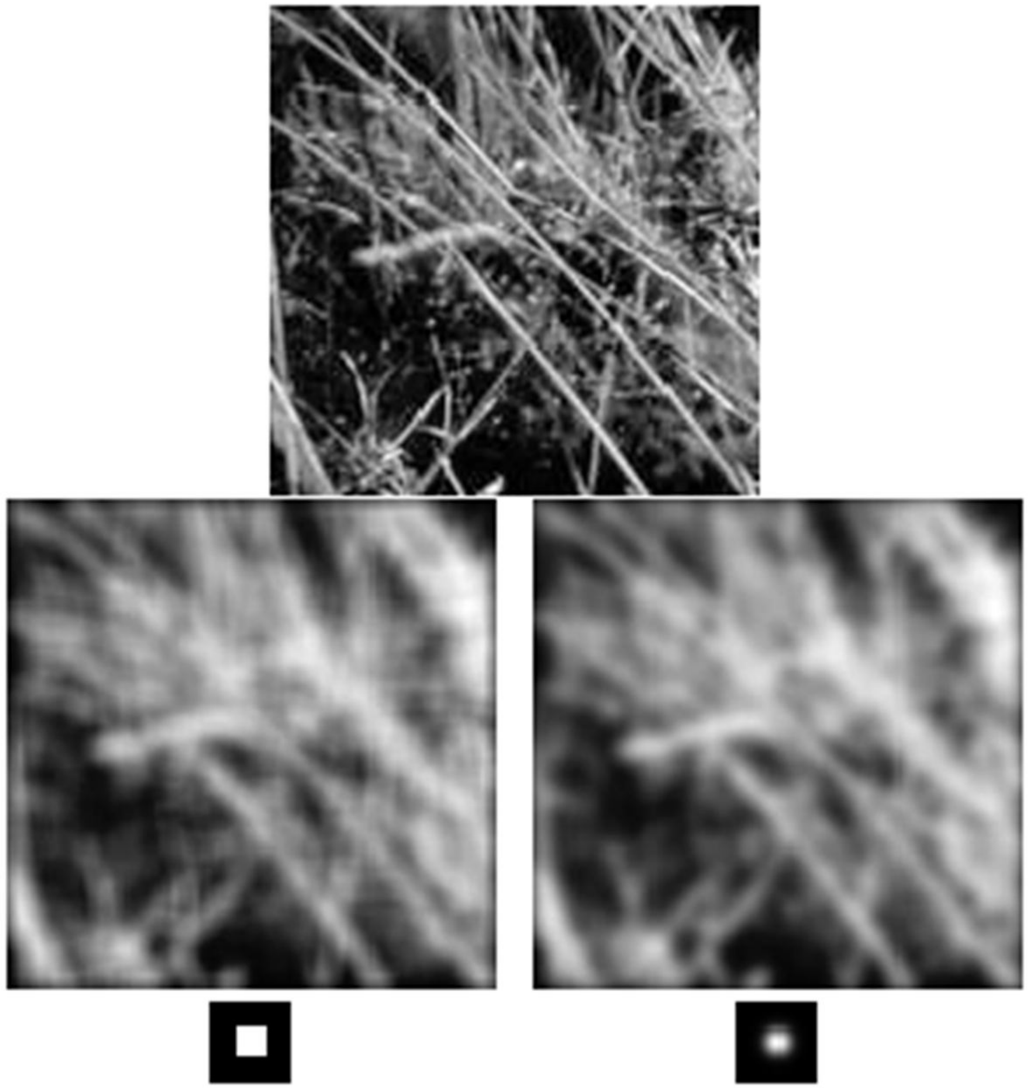
Gaussian filtering





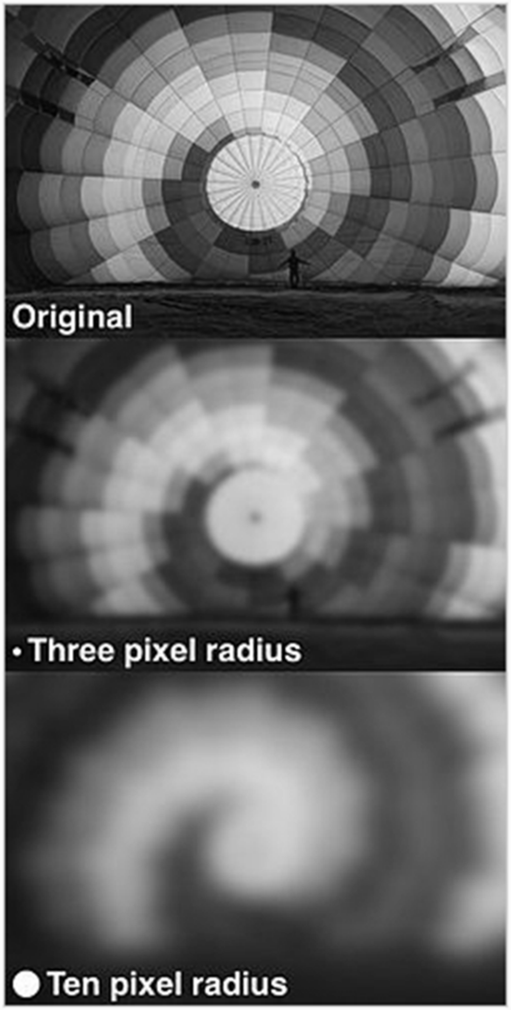
Gaussian filter, 20x20, $\sigma = 5$

Mean vs. Gaussian filtering



Neighborhood Processing (filtering)

Gaussian Filtering



The effects of a small and a large Gaussian blur

- Low-pass filter
- Smooth color variation (low frequency) is preserved
- Sharp edges (high frequency) are removed

Median filters

- A **Median Filter** operates over a window by selecting the median intensity in the window.

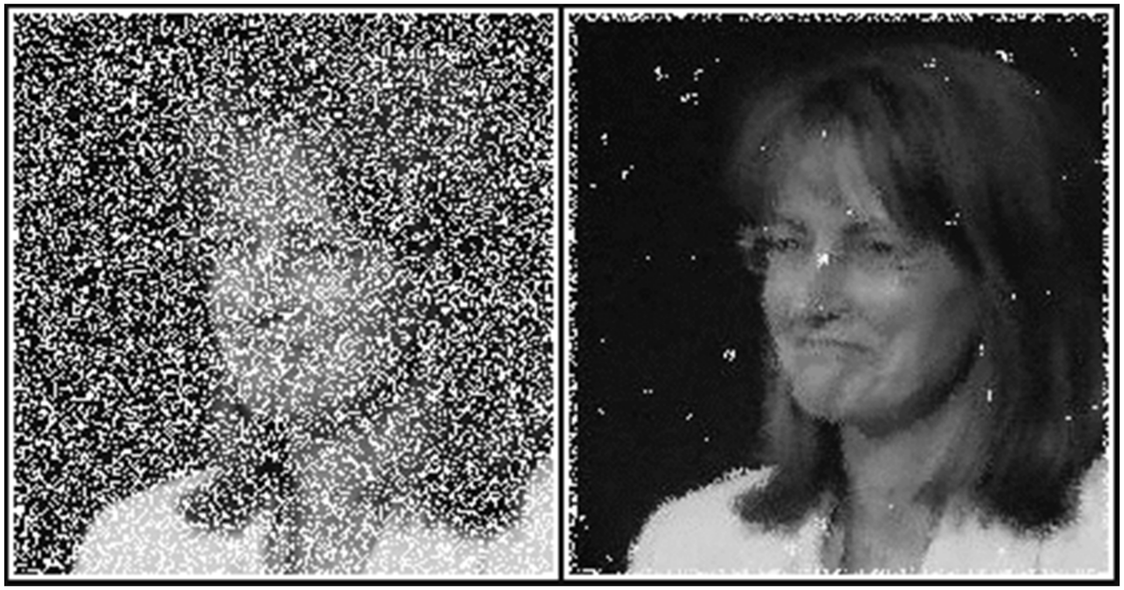


Image credit: Wikipedia – page on Median Filter

Median filters

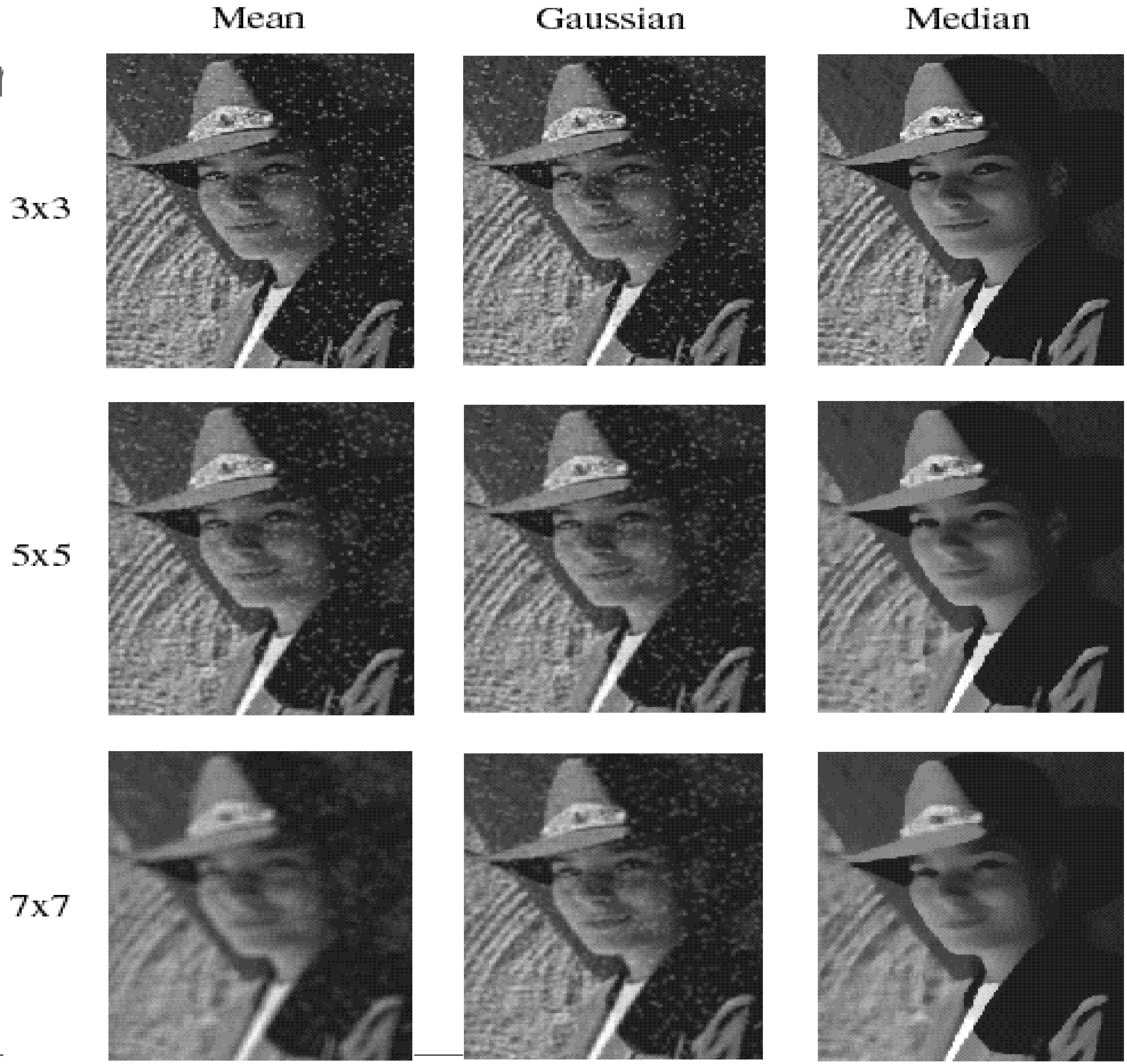
- Is a median filter a kind of convolution?
  No, median filter is an example of non-linear filtering

$$Median(f_1(x)+f_2(x)) \neq Median(f_1(x)) + Median(f_2(x))$$

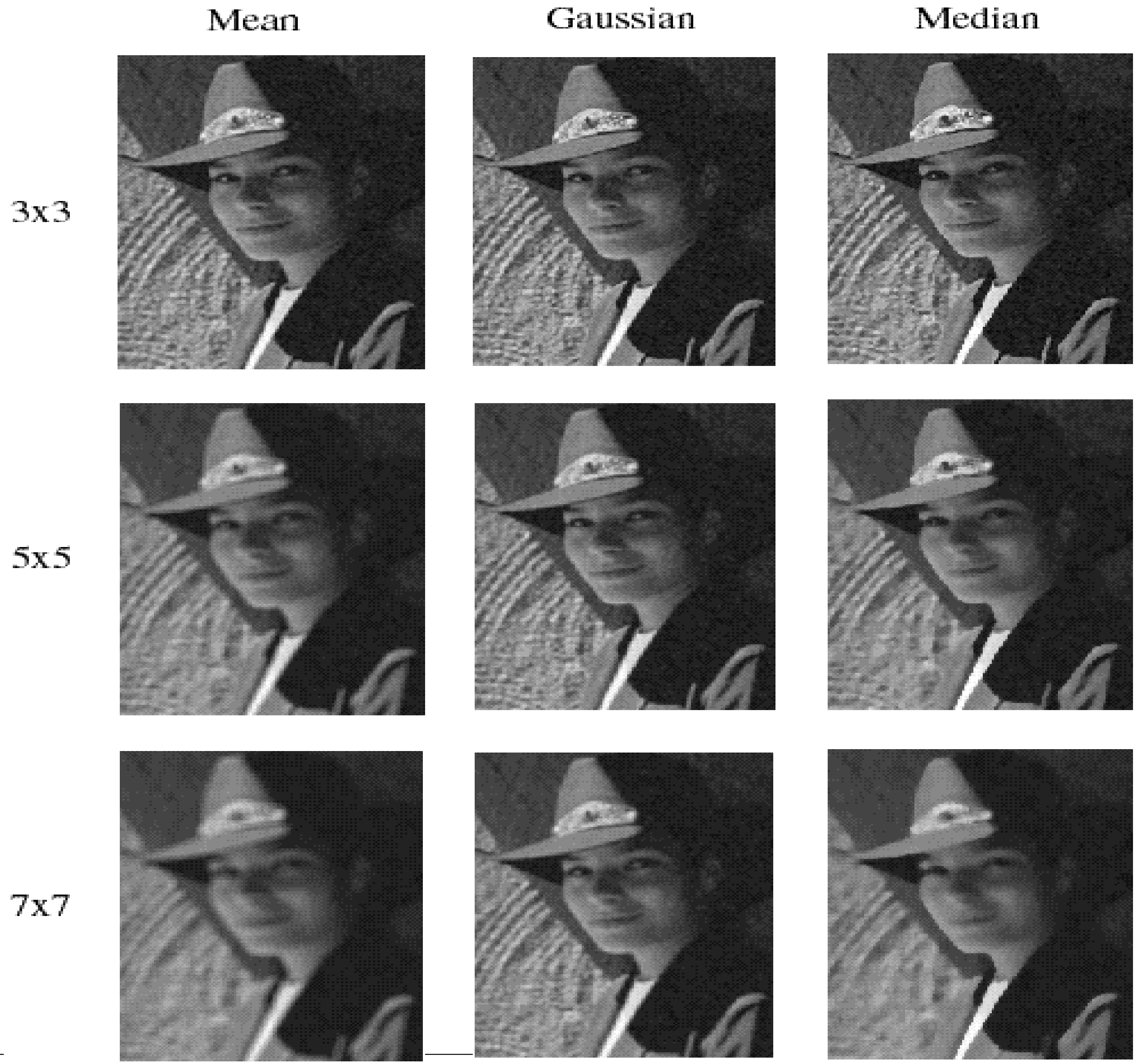
$$1 = Median(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}) = Median(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}) \neq$$

$$Median(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}) + Median(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}) = 1+1 = 2$$

Salt&Pepper Noise



Gaussian noise



Median filters

- What advantage does a **median filter** have over a **mean filter**?
  Better at removing Salt & Pepper noise
- Disadvantage:
  Slow

Neighborhood Processing (filtering)

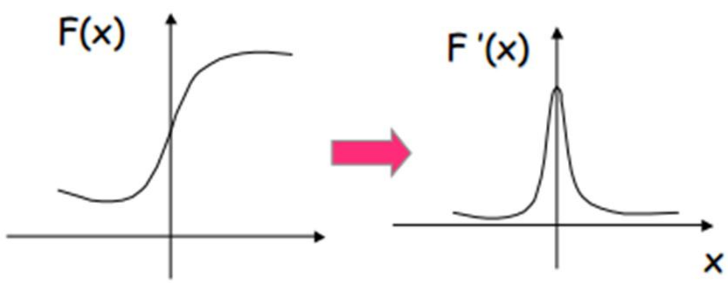
Derivatives and Convolution

- Image Derivatives and Gradients
  - Used for Edge / Corner Detection
  - Computed with Finite Differences Filters
- Laplacian of Gaussians (LoG) Filter
  - Used for Edge / Blob Detection and Image Enhancement
  - Approximated using Difference of Gaussians

Reading: Forsyth & Ponce, 8.1-8.2

First Derivative

- Recall Sharp changes in gray level of the input image correspond to “peaks or valleys” of the first-derivative of the input signal.



(1D example)

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6pp.

Reading: Forsyth & Ponce, 8.1-8.2

First Derivative and convolution

$$\frac{\partial}{\partial x} f = \lim_{h \to 0} \left(\frac{f(x+h, y) - f(x, y)}{h} \right)$$

- How can we approximate it for a discrete function?
- Is this operation shift-invariant?
- Is it linear?

?	?	?
?	?	?
?	?	?

$\nabla_x [u, v]$

Reading: Forsyth & Ponce, 8.1-8.2

First Derivative and convolution

- Finite Difference

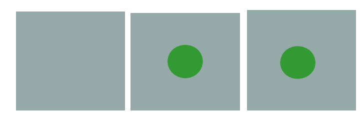
$$f(x+a) - f(x+b)$$

Forward, backward, and central differences

Only three forms are commonly considered: forward, backward, and central differences.

A **forward difference** is an expression of the form

$$\Delta_h[f](x) = f(x+h) - f(x).$$



$$\frac{\Delta_h[f]}{h} = \frac{f(x+h) - f(x)}{h} \xrightarrow[h \to 0]{} f'(x)$$

Slide credit: Wikipedia

Reading: Forsyth & Ponce, 8.1-8.2

First Derivative and convolution

- Finite Difference – The order of an error can be derived using Taylor Theorem

$$\frac{\Delta_h[f](x)}{h} - f'(x) = O(h) \quad (h \to 0).$$

The same formula holds for the backward difference:

$$\frac{\nabla_h[f](x)}{h} - f'(x) = O(h).$$

However, the central difference yields a more accurate approximation.

$$\frac{\delta_h[f](x)}{h} - f'(x) = O(h^2).$$

Slide credit: Wikipedia

Reading: Forsyth & Ponce, 8.1-8.2

First Derivative and convolution

- Pixel Size $\Delta x = h$
- Using Finite Central Difference

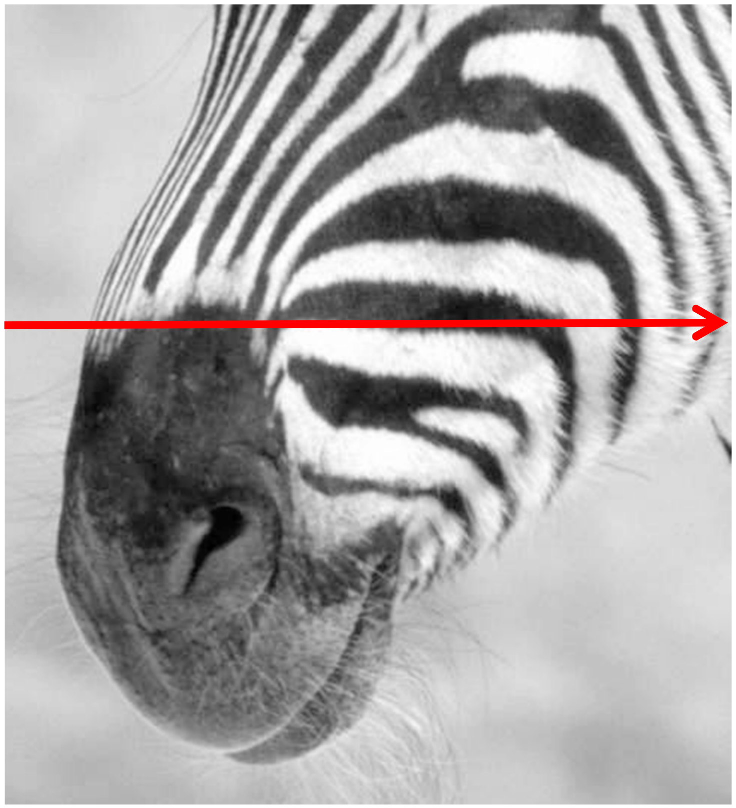
$$\frac{\partial}{\partial x} f \approx \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2 \cdot \Delta x}$$

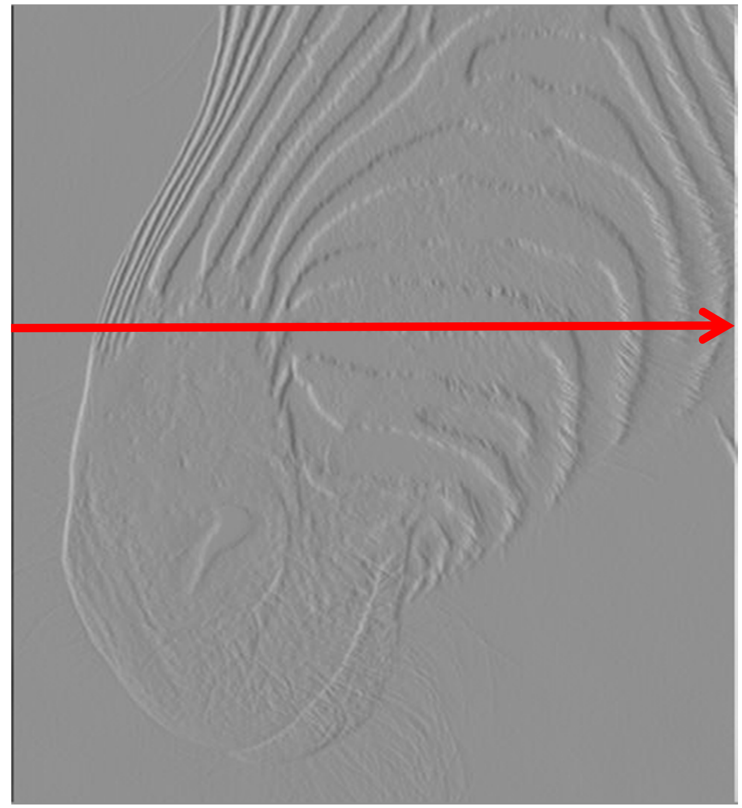
- We need a kernel ∇_x such that

$$\frac{\partial}{\partial x} f = \nabla_x * f$$

$$\nabla_x[u,v] = \frac{1}{2\Delta x} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Finite differences – responds to edges





Dark = negative

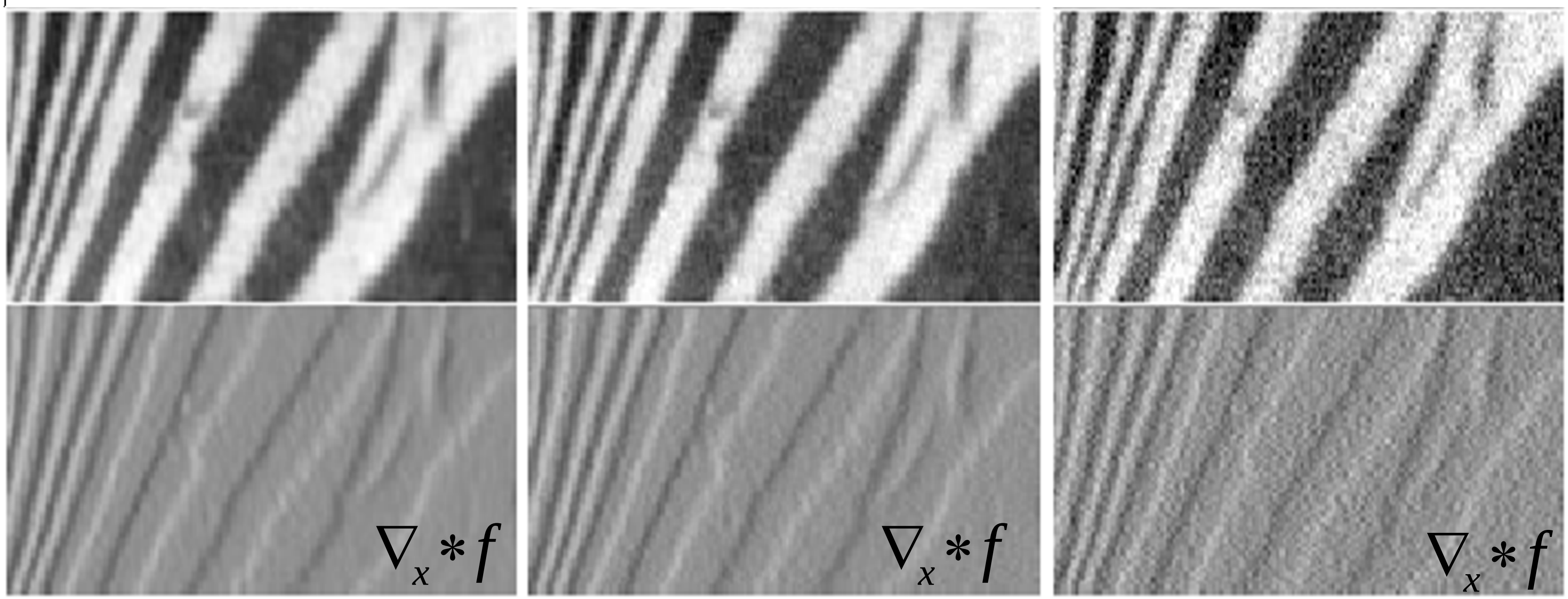
White = positive

Gray = 0

$$\nabla_x * f$$

Neighborhood Processing (filtering)

Finite differences - responding to noise



Increasing noise ->

(this is zero mean additive gaussian noise)

Finite differences and noise

- Finite difference filters respond strongly to noise
  - Noisy pixels look very different from their neighbours
  - The larger the noise ➔ the stronger is the response

- How can we eliminate the response to noise?
  - Most pixels in images look similar to their neighbours (even at an edge)
  - Smooth the image (mean/gaussian filtering)

Neighborhood Processing (filtering)

Smoothing and Differentiation

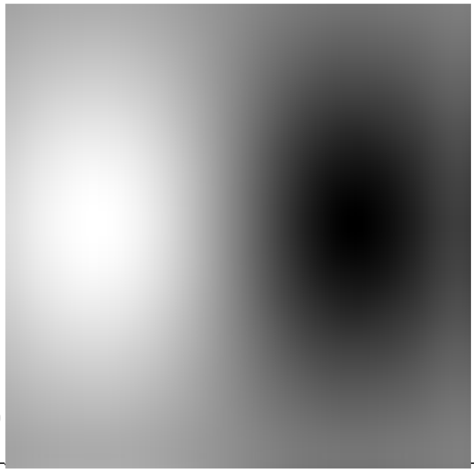
- Smoothing before differentiation = two convolutions

$$\nabla_x * (H * f)$$

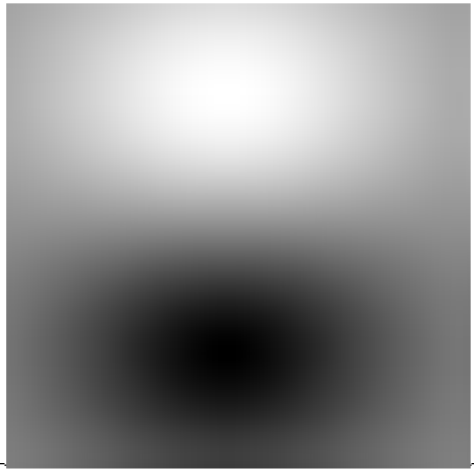
- Convolution is associative

$$\nabla_x * (H * f) = (\nabla_x * H) * f$$

$(\nabla_x * H)$



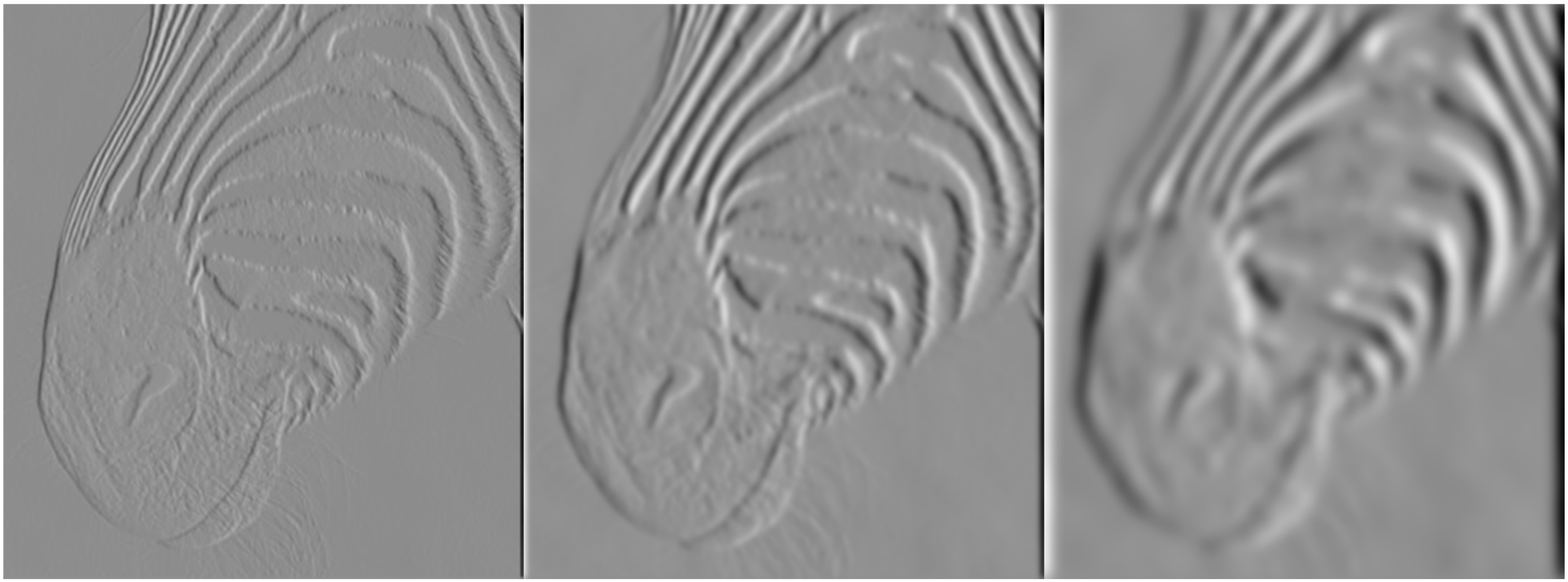
$(\nabla_y * H)$



Neighborhood Processing (filtering)

Smoothing and Differentiation

$$(\nabla_x * H) * f$$



1 pixel 3 pixels 7 pixels

The scale of the smoothing filter affects derivative estimates, and also the semantics of the edges recovered.

Sobel kernels

- Yet another approximation frequently used

$$\frac{\partial}{\partial x} f \qquad \frac{\partial}{\partial y} f$$

$$\frac{1}{8\Delta x} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \qquad \frac{1}{8\Delta y} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$$\nabla_x[u,v] \qquad \nabla_y[u,v]$$

Image Gradients

- Recall for a function of two variables $f(x,y)$
- The **gradient** at a point (x,y)

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \approx \begin{bmatrix} \nabla_x * f \\ \nabla_y * f \end{bmatrix}$$

- **Gradient Magnitude**

$$\| \nabla f \| = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} \approx$$

- **Gradient Orientation**
  - direction of the "steepest ascend"
  - orthogonal to object boundaries in the image

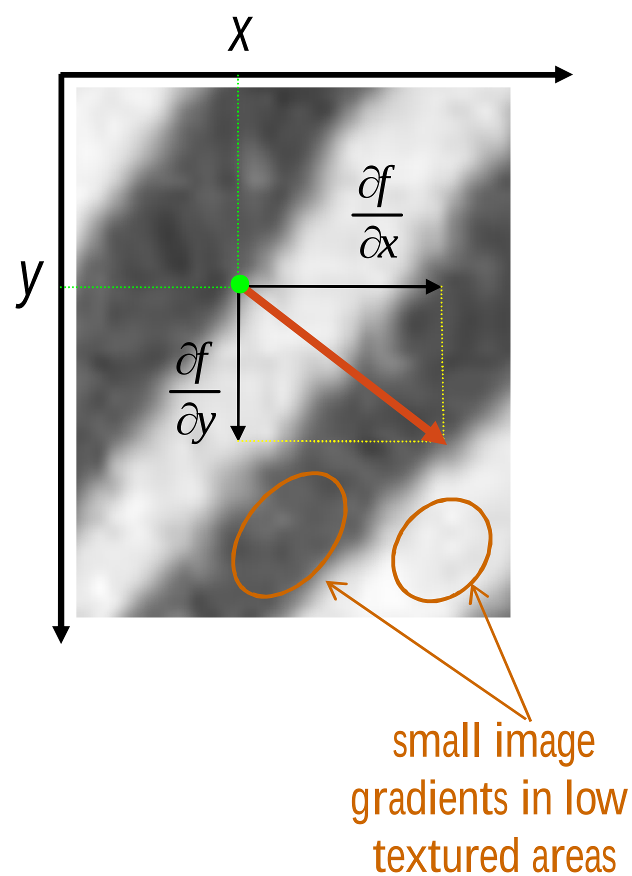


Image Gradient for Edge Detection

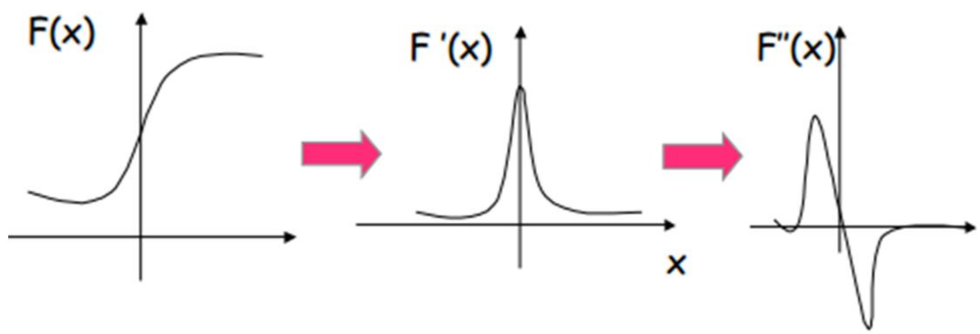
- Typical application where image gradients are used is *image edge detection*
  - find points with large image gradients

Canny edge detector suppresses non-extrema Gradient points

Reading: Forsyth & Ponce, 8.1-8.2

Second derivatives and convolution

- Peaks or valleys of the first-derivative of the input signal, correspond to "zero-crossings" of the second-derivative of the input signal.



http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6pp.

Second derivatives and convolution

Taylor Series expansion

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{3!}h^3 f'''(x) + O(h^4)$$

add

$$+ \left[f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{3!}h^3 f'''(x) + O(h^4) \right]$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

$$\frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x) + O(h^2)$$

1	-2	1

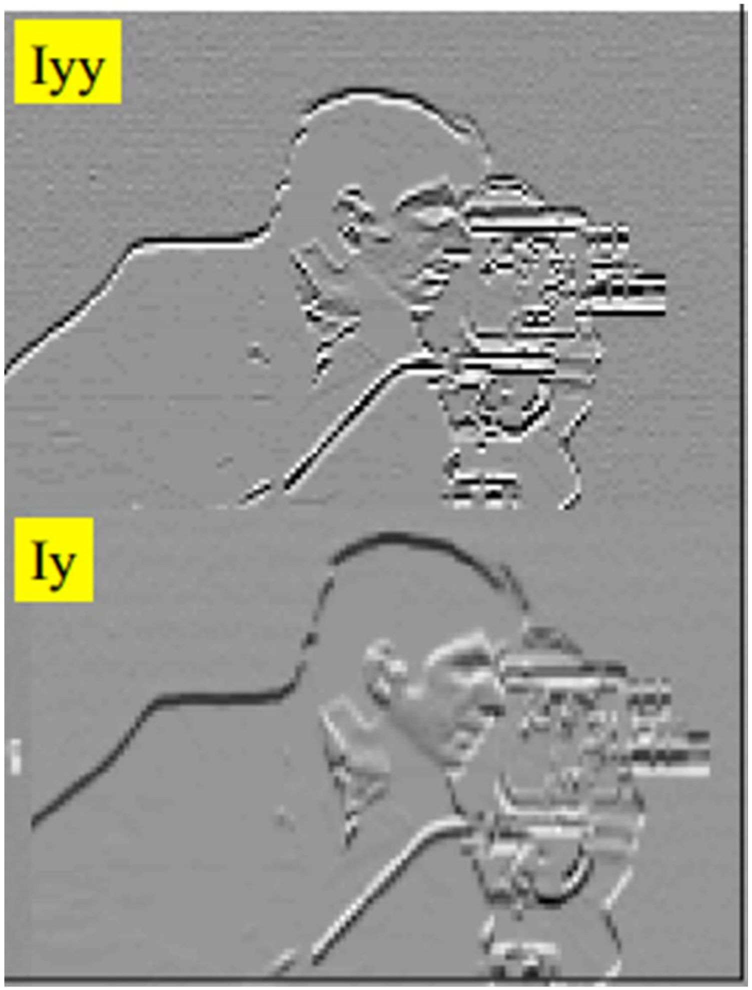
Central difference approx to second derivative

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6pp.

Neighborhood Processing (filtering)

Second derivatives and convolution

- Better localized edges
- But more sensitive to noise



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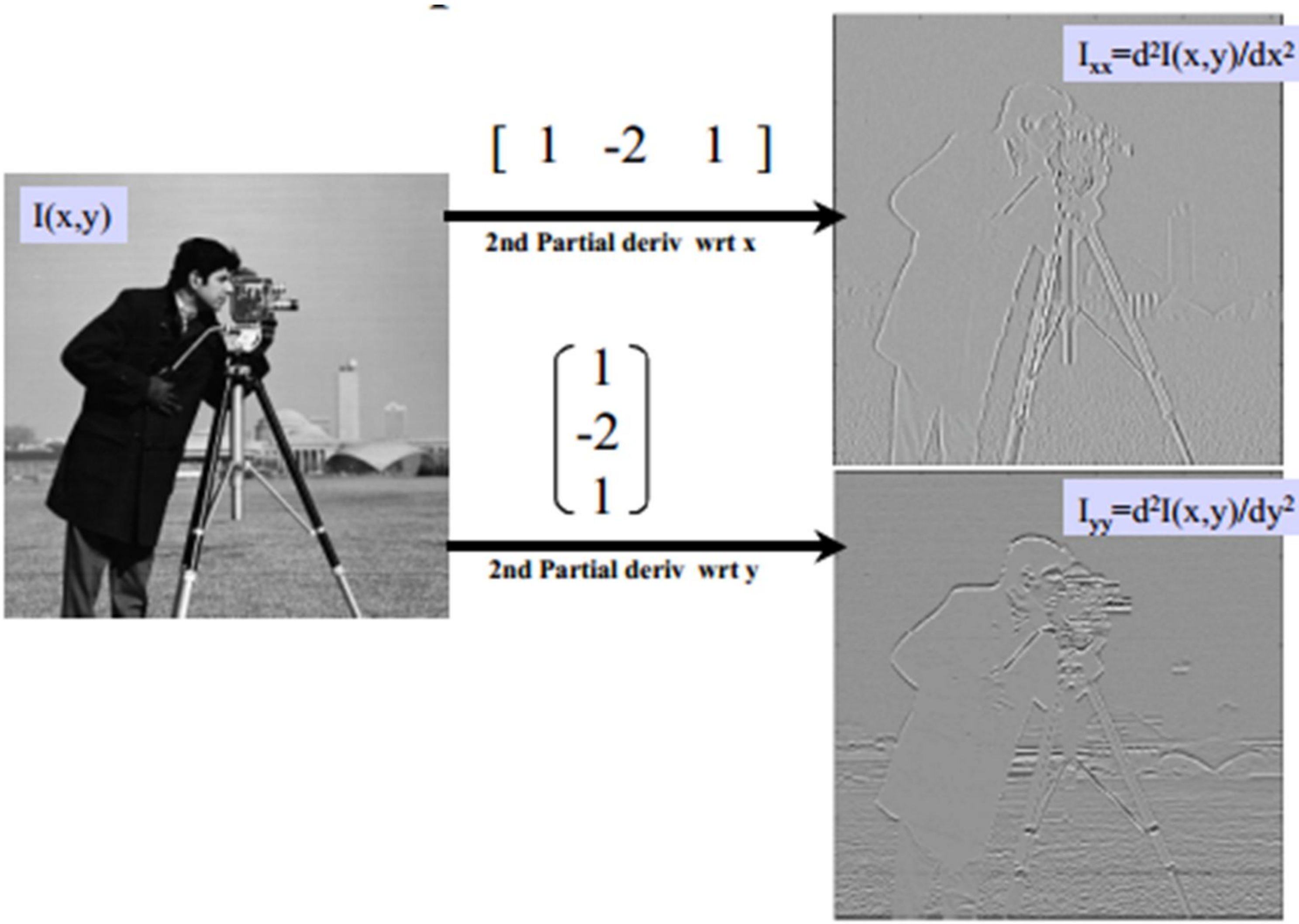
Neighborhood Processing (filtering)

Second derivatives and convolution



http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Neighborhood Processing (filtering)

Second Image Derivatives

- Laplace operator

$$\Delta f = \nabla \cdot \nabla f = \nabla^2 f$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

rotationally invariant
second derivative for 2D functions

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Finite Difference Second Order Derivative in x

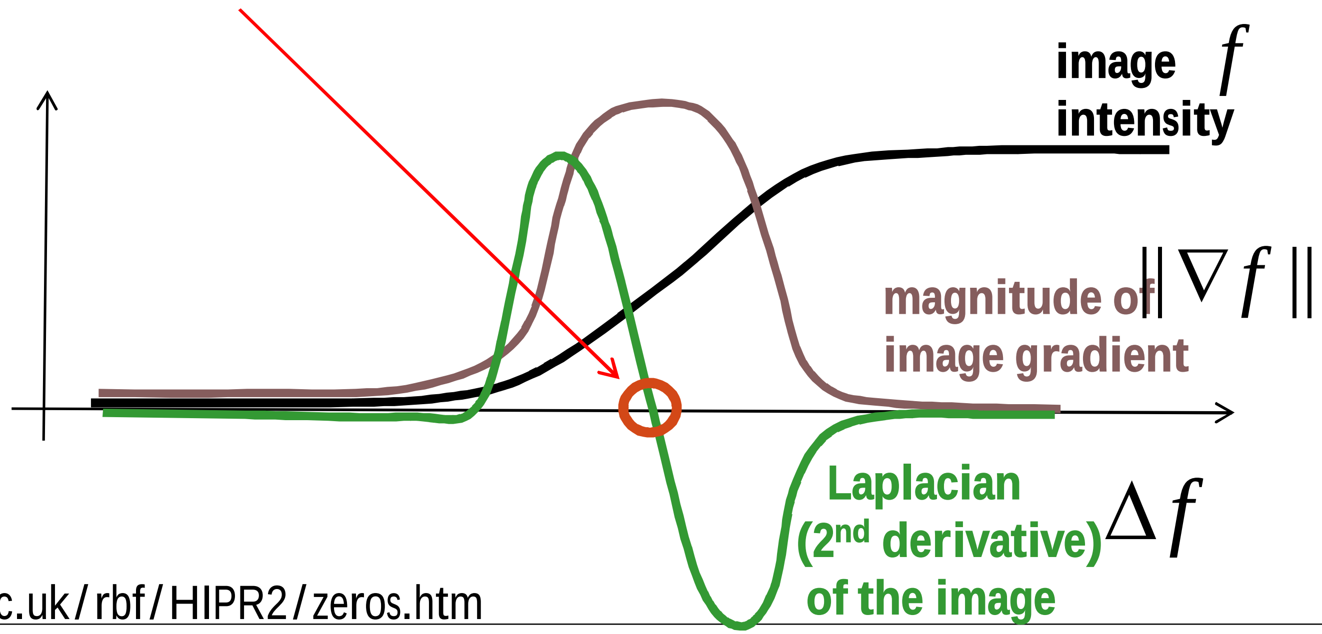
Finite Difference Second Order Derivative in y

Second Image Derivatives

- Laplacian Zero Crossing

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

- Used for edge detection
  (alternative to computing Gradient extrema)



http://homepages.inf.ed.ac.uk/rbf/HIPR2/zeros.htm

Neighborhood Processing (filtering)

Laplacian Filtering

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} * I$$

-Zero on uniform regions
-Positive on one side of an edge
-Negative on the other side
-Zero at some point in between
on the edge itself
➔ band-pass filter (Suppresses both high and low frequencies)

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

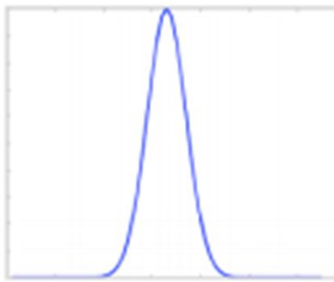
Neighborhood Processing (filtering)

Laplacian of a Gaussian (LoG)

- Smooth before differentiation
  (remember associative property of convolution)

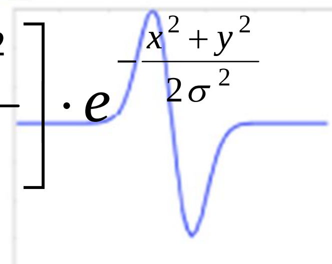
$$\Delta * G = LoG$$

$$g(x) = e^{-\frac{x^2}{2\sigma^2}}$$

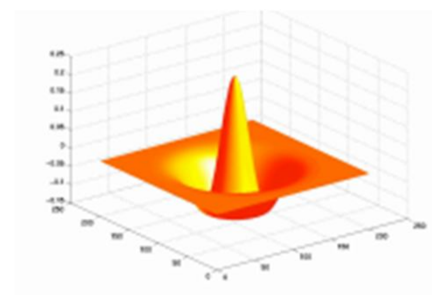


$$g'(x) = -\frac{1}{2\sigma^2} 2x e^{-\frac{x^2}{2\sigma^2}} = -\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

$$LoG(x, y) = -\frac{1}{\pi\sigma^4}\left[1 - \frac{x^2 + y^2}{2\sigma^2}\right] \cdot e^{-\frac{x^2 + y^2}{2\sigma^2}}$$



$$g''(x) = \left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right) e^{-\frac{x^2}{2\sigma^2}}$$

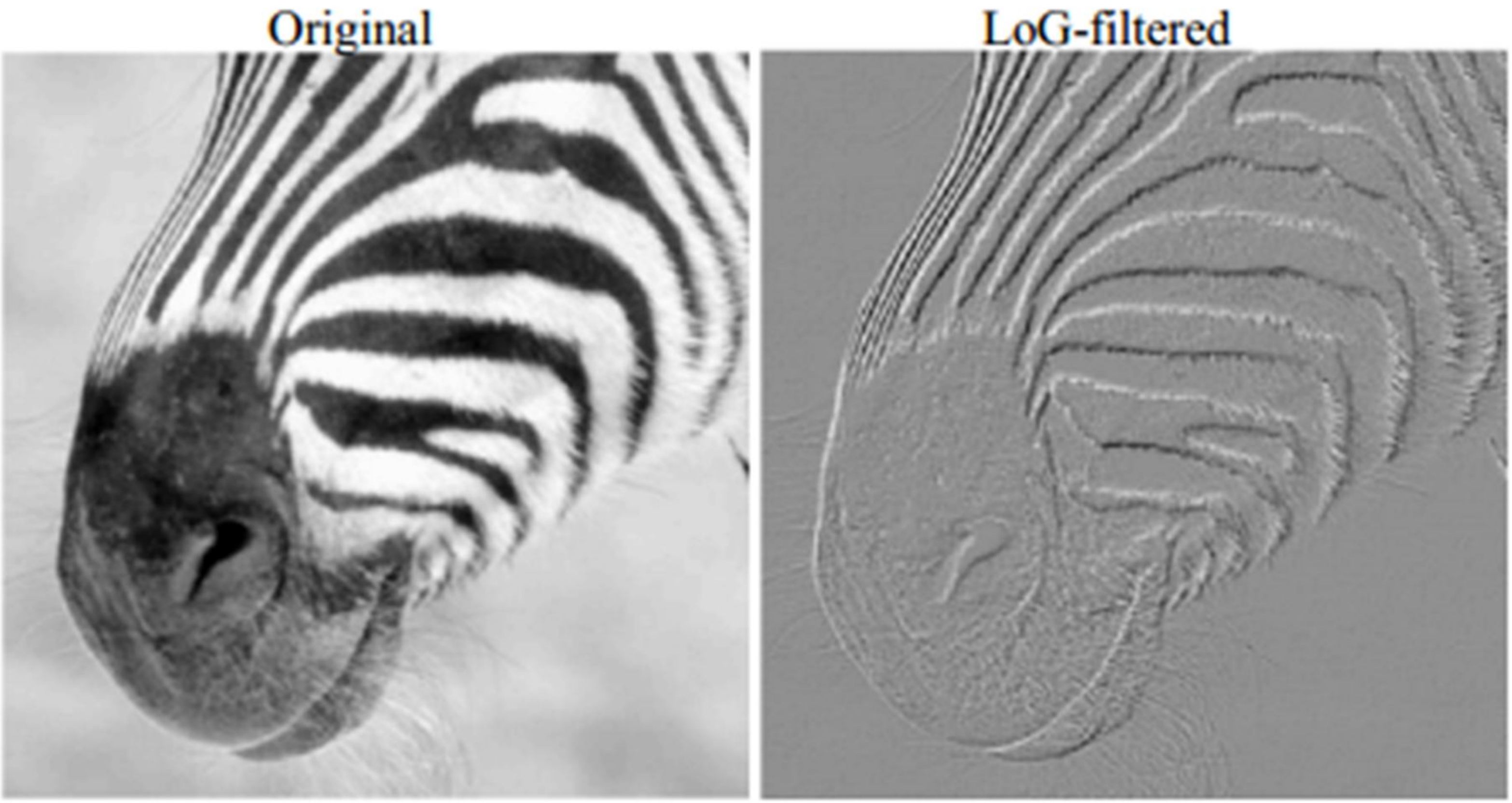


LoG "Mexican Hat"

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Neighborhood Processing (filtering)

Laplacian of a Gaussian (LoG)



Suppresses both high and low frequencies

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Laplacian of a Gaussian (LoG)

Raw zero-crossings (no contrast thresholding)

LoG sigma = 2, zero-crossing

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Laplacian of a Gaussian (LoG)

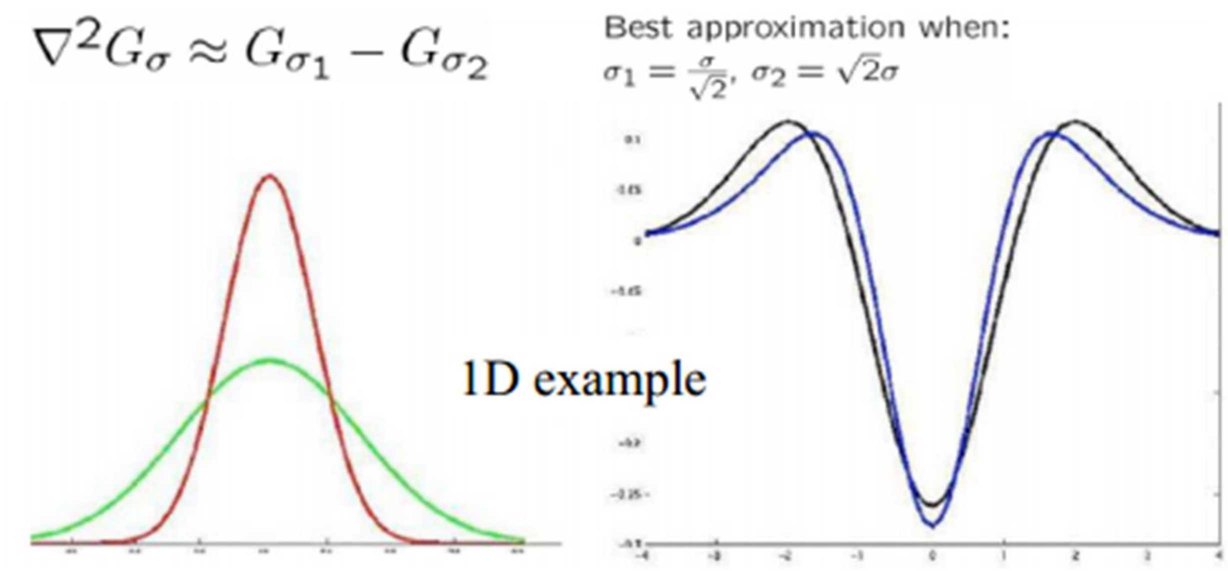
Raw zero-crossings (no contrast thresholding)

LoG sigma = 4, zero-crossing

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Laplacian of a Gaussian (LoG)

- Can be approximated by a difference of two Gaussians (DoG)



http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Neighborhood Processing (filtering)

DoG vs. LoG

- Separable (product decomposition) ➔ more efficient
- Can explain band-pass filter since Gaussian is a low-pass filter.

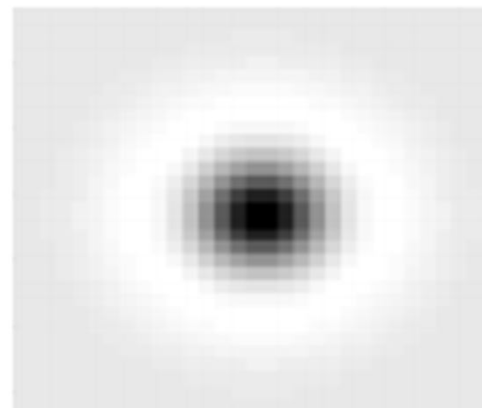
http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Neighborhood Processing (filtering)

LoG for Blob Detection

- Cross correlation with a filter can be viewed as comparing a little "picture" of what you want to find against all local regions in the image.

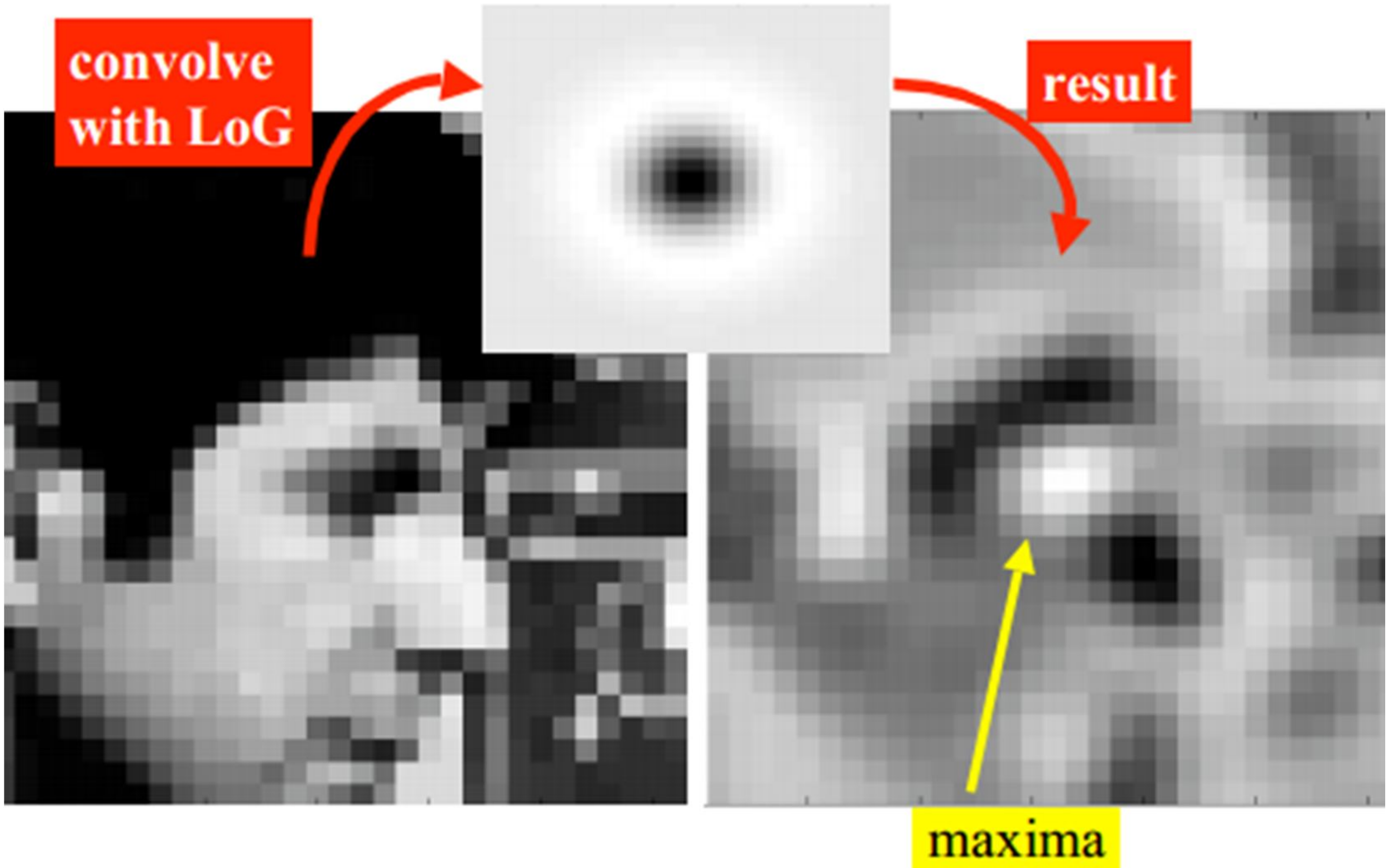
- Scale of blob (size ; radius in pixels) is determined by the sigma parameter of the LoG filter.



Maximum response:
dark blob on light background
Minimum response:
light blob on dark background

http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

LoG for Blob Detection



http://www.cse.psu.edu/~rcollins/CSE486/lecture11_6p

Histogram Equalization

r_k	n_k	$p_r(r_k) = n_k/MN$	$7 * \sum_u p(u)$	v
$r_0 = 0$	790	0.19	1.33	1
$r_1 = 1$	1023	0.25	3.08	3
$r_2 = 2$	850	0.21	4.55	5
$r_3 = 3$	656	0.16	5.67	6
$r_4 = 4$	329	0.08	6.23	6
$r_5 = 5$	245	0.06	6.65	7
$r_6 = 6$	122	0.03	6.86	7
$r_7 = 7$	81	0.02	7.00	7