FILTERING IN THE FREQUENCY DOMAIN
Spatial Vs Frequency domain

- **Spatial Domain (I)**
  - “Normal” image space
  - Changes in pixel positions correspond to changes in the scene
  - Distances in I correspond to real distances

- **Frequency Domain (F)**
  - Changes in image position correspond to changes in the spatial frequency
  - This is the rate at which image intensity values are changing in the spatial domain image I
Periodic functions can be expressed as the sum of sines and/or cosines of different frequencies each multiplied by a different coefficient.
Image processing

- **Spatial Domain (I)**
  - Directly process the input image pixel array

- **Frequency Domain (F)**
  - Transform the image to its frequency representation
  - Perform image processing
  - Compute inverse transform back to the spatial domain
Frequencies in an Image

- Any spatial or temporal signal has an equivalent frequency representation.

- What do frequencies mean in an image?
  - High frequencies correspond to pixel values that change rapidly across the image (e.g. text, texture, leaves, etc.)
  - Strong low frequency components correspond to large scale features in the image (e.g. a single, homogenous object that dominates the image)

- We will investigate Fourier transformations to obtain frequency representations of an image.
Properties of a Transform

A transform maps image data into a different mathematical space via a transformation equation.

Most of the discrete transforms map the image data from the spatial domain to the frequency domain, where all the pixels in the input (spatial domain) contribute to each value in the output (frequency domain).
Spatial Frequency

- Rate of change

- Faster the rate of change over distance, higher the frequency
Image Transforms

Image transforms are used as tools in many applications, including enhancement, restoration, correlation and SAR data processing.

Discrete Fourier transform is the most important transform employed in image processing applications.

Discrete Fourier transform is generated by sampling the basis functions of the continuous transform i.e., the sine and cosine functions.
The Fourier transform decomposes a complex signal into a weighted sum of sinusoids, starting from zero-frequency to a high value determined by the input function.

The lowest frequency is also called the fundamental frequency.
The base frequency or the fundamental frequency is the lowest frequency. All multiples of the fundamental frequency are known as harmonics.

A given signal can be constructed back from its frequency decomposition by a weighted addition of the fundamental frequency and all the harmonic frequencies
Different forms of Fourier Transform

Continuous Fourier Transform

\[
F(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} \, dx
\]

Fourier Series

\[
f(x) = a_0 + \sum_{n=-\infty}^{+\infty} a_n \cos(2\pi nx) + b_n \sin(2\pi nx)
\]

where

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(2\pi nx) \, dx
\]

\[
b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(2\pi nx) \, dx
\]
Continuous Fourier Transform

- In the continuous domain, the basis functions of the Fourier transform are the complex exponentials $e^{-j2\piux}$

- These functions extend from $-\infty$ to $+\infty$

- These are continuous functions, and exist everywhere
Real and Imaginary Parts of Fourier Transform

\[ F(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} \, dx \]

\[ F(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) \cos(2\pi ux) \, dx - \frac{1}{2\pi} j \int_{-\infty}^{+\infty} f(x) \sin(2\pi ux) \, dx \]

Real part  Imaginary Part
The Discrete Fourier Transform

- Since we are dealing with images, we will be more interested in the discrete Fourier Transform (DFT)

- For a function $f(x)$, $x=0,1,\ldots,M-1$ we have

$$
F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}, \quad u = 0,\ldots,M-1
$$

$$
iDFT \quad f(x) = \sum_{x=0}^{M-1} F(u)e^{j2\pi ux/M}, \quad x = 0,\ldots,M-1
$$

- For discrete functions, the DFT and iDFT always exist
The Discrete Fourier Transform

- Recall Euler’s Formula

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

from which we obtain

\[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \left( \cos \frac{2\pi ux}{M} - j \sin \frac{2\pi ux}{M} \right) \]

for \( u = 0, \ldots, M-1 \)

- Each term is composed of ALL values of \( f(x) \)
- The values of \( u \) are the frequency domain
- Each \( F(u) \) is a frequency component of the transform
The 2-D Discrete Fourier Transform

- Since our images are nothing more than 2-D discrete functions, we are interested in the 2-D DFT

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} \]

for \( u=0,\ldots,M-1 \) and \( v=0,\ldots,N-1 \) and the iDFT is defined as

\[ f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)} \]

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The 2-D Discrete Fourier Transform

Since the values of the FT are complex numbers, it is sometimes more convenient to express $F(u,v)$ in terms of polar coordinates

$$F(u,v) = |F(u,v)| e^{-j\phi(u,v)}$$

where the magnitude or spectrum is denoted by

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

and the phase angle by

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$
The 2-D Discrete Fourier Transform

DFT (Continued)

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} \]

- At \( u=v=0 \), the FDT reduces to

\[ F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \]

- This is nothing more than the average grayscale level of the image

- This is often referred to as the DC Component (0 frequency)
The 2-D Discrete Fourier Transform

DFT (Continued)

- The FT has the following translation property

\[ f(x, y) e^{-j2\pi(\frac{u_0 x}{M} + \frac{v_0 y}{N})} \Leftrightarrow F(u-u_0, v-v_0) \]

which for \( u_0 = M/2 \) and \( v_0 = N/2 \) we see that

\[ e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} = e^{-j\pi(x+y)} = (-1)^{x+y} \]

\[ \Rightarrow F(f(x, y)(-1)^{x+y}) = F(u-M/2, v-N/2) \]

- In image processing, it is common to multiply the input image by \((-1)^{x+y}\) prior to computing \( F(u,v) \)

- This has the effect of centering the transform since \( F(0,0) \) is now located at \( u=M/2, \ v=N/2 \)
Properties of the Fourier Transform

- The FT is a linear operator
  \[ F(af + bg) = aF(f) + bF(g) \]

- Some other useful properties include

  ![Diagram with properties]

  Basis for filtering in the frequency domain!
Filtering Example
Smooth an Image with a Gaussian Kernel

- Traditionally, we would just convolve the image with the a gaussian kernel

- Instead, we will perform multiplication in the frequency domain to achieve the same effect
Filtering Example
Smooth an Image with a Gaussian Kernel

1. Multiply the input image by \((-1)^{x+y}\) to center the transform
Filtering Example
Smooth an Image with a Gaussian Kernel

2. Compute the DFT $F(u,v)$ of the resulting image
Filtering Example
Smooth an Image with a Gaussian Kernel

3. Multiply $F(u,v)$ by a filter $G(u,v)$

$$g(x,y) = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256$$

$G(u,v)$
Filtering Example
Smooth an Image with a Gaussian Kernel

3. Multiply $F(u,v)$ by a filter $G(u,v)$
Filtering Example
Smooth an Image with a Gaussian Kernel

4. Computer the inverse DFT transform $h^*(x,y)$
5. Obtain the real part $h(x,y)$ of 4
6. Multiply the result by $(-1)^{x+y}$
The Fourier Transform

- Functions that are NOT periodic BUT with finite area under the curve can be expressed as the integral of sines and/or cosines multiplied by a weight function

\[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \, dx \]

\[ f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} \, du \]
Properties of the Fourier Transform

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- Some other useful properties include

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Some Fundamental Transform Pairs

\[ F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux} \, dx = \int_{-1}^{1} e^{-j2\pi ux} \, dx \]

\[ = \frac{-1}{j2\pi u} (e^{j2\pi u} - e^{-j2\pi u}) = \frac{\sin 2\pi u}{\pi u} \]
Some Fundamental Transform Pairs

\[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi u x} dx = \int_{-\infty}^{\infty} e^{-ax^2} \left( \cos 2\pi u x - j \sin 2\pi u x \right) dx \]

\[ = \int_{-\infty}^{\infty} e^{-ax^2} \cos 2\pi u x dx - j \int_{-\infty}^{\infty} e^{-ax^2} \sin 2\pi u x dx \]

\[ = \sqrt{\frac{\pi}{a}} e^{-\pi^2 u^2 / a} \]
Example

Given \( f(n) = [3, 2, 2, 1] \), corresponding to the brightness values of one row of a digital image. Find \( F(u) \) in both rectangular form and in exponential form.
Example Contd.

\[ F(0) = \frac{1}{4} [3 + 2 + 2 + 1] = 2 \]

\[ F(1) = \frac{1}{4} [3 + 2e^{-j2\pi 1.1/4} + 2e^{-j2\pi 2.1/4} + 1.e^{-j2\pi 3.1/4}] = \]

\[ \frac{1}{4} [3 + 2 - 2j + j] = \frac{1}{4} [1 - j] \]
Example Contd.

\[ F(2) = \frac{1}{4}[3 - 2 + 2 - 1] = \frac{1}{2} \]

\[ F(3) = \frac{1}{4}[3 + 2e^{-j\frac{2\pi}{3.1/4}} + 2e^{-j\frac{2\pi}{3.2/4}} + 1.e^{-j\frac{2\pi}{3.3/4}}] = \]

\[ \frac{1}{4}[3 + 2j - 2 - j] = \frac{1}{4}[1 + j] \]

Therefore \( F(u) = [2 \quad \frac{1}{4}(1-j) \quad \frac{1}{2} \quad \frac{1}{4}(1+j)] \)
Magnitude-Phase Form

\[ F(0) = 2 = 2 + j0 \Rightarrow \text{Mag} = \sqrt{2^2 + 0^2} = 2; \quad \text{Phase} = \tan^{-1}(0/2) = 0 \]

\[ F(1) = \frac{1}{4} (1-j) = \frac{1}{4} - j \frac{1}{4} \Rightarrow \text{Mag} = \frac{1}{4} \sqrt{1^2 + (-1)^2} = 0.35; \]
\[ \text{Phase} = \tan^{-1}(-1/4 / 1/4) = \tan^{-1}(-1) = -\pi/4 \]

\[ F(2) = \frac{1}{2} = \frac{1}{2} + j0 \Rightarrow \text{Mag} = \sqrt{(\frac{1}{2})^2 + 0^2} = \frac{1}{2} \]
\[ \text{Phase} = \tan^{-1}(0 / (1/2)) = 0 \]

\[ F(3) = \frac{1}{4} (1+j) = \frac{1}{4} - j \frac{1}{4} \Rightarrow \text{Mag} = \frac{1}{4} \sqrt{1^2 + (-1)^2} = 0.35; \]
\[ \text{Phase} = \tan^{-1}(1/4 / 1/4) = \tan^{-1}(1) = \pi/4 \]
Given $f(n) = [3 \ 2 \ 2 \ 1]$

$F(u) = [2 \ \frac{1}{4} (1-j) \ \frac{1}{2} \ \frac{1}{4} (1+j)]$

In phase magnitude form

$M(u) = [2 \ 0.35 \ \frac{1}{2} \ 0.35]$ 

$\Phi(u) = [0 \ -\frac{\pi}{4} \ 0 \ \frac{\pi}{4}]$

Calculate the above for $f(n) = [0 \ 0 \ 4 \ 4 \ 4 \ 4 \ 0 \ 0]$

Plot $f(n)$, $F(u)$, $M(u)$ and $\Phi(u)$ graphically